Enumerative geometry and Virasoro constraints

Miguel Moreira
MIT

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UMass Boston
Mathematics Colloquium

Enumerative geometry

In enumerative geometry the goal is to count how many geometric objects satisfy certain restrictions. A model question is the following:

Question

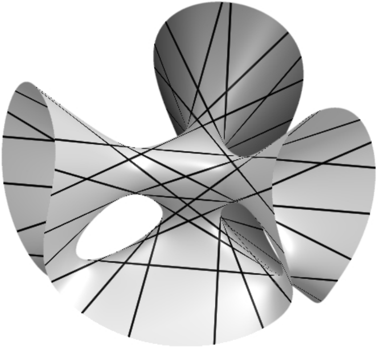
Let $f(x, y, z, w)$ be a homogeneous cubic polynomial and

$$X = \{[x : y : z : w] : f(x, y, z, w) = 0\} \subseteq \mathbb{C}P^3$$

be a (smooth) cubic surface. How many lines does X contain?

27 lines on the Fermat cubic

The Fermat cubic is defined by $x^3 + y^3 + z^3 + w^3 = 0$.



It contains 27 lines, parametrized by $[x : \omega_1 x : z : \omega_2 z]$ where $\omega_1^3 = \omega_2^3 = -1$ (9 possibilities) or permutations of the coordinates (3 possibilities).

27 lines

Theorem (Cailey–Salmon, 1849)

Every smooth cubic surface contains exactly 27 lines.

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- To prove this, we want to look at the space of all lines in $\mathbb{C}P^3$ and then try to figure out how many of them are contained in X using intersection theory.
- A line on $\mathbb{C}P^3$ is the same as a 2-dimensional vector subspace of $\mathbb{C}^4$, so the space of lines on $\mathbb{C}P^3$ is an example of a Grassmannian $\mathrm{Gr}(\mathbb{C}^4, 2)$.

Grassmannians

Definition

The Grassmannian $\mathrm{Gr}(\mathbb{C}^N, k)$ is the space of k dimensional subspaces of $\mathbb{C}^N$. Explicitly, it can be defined as a quotient

$$\mathrm{Gr}(\mathbb{C}^N, k) = \{A \in \mathrm{Mat}_{k\times N} \colon \mathrm{rk}(A) = k\}/GL(k)\,.$$

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- Grassmannians are arguably the simplest examples of moduli spaces. Moduli space: a space whose points correspond to "geometric" objects (in this case, vector subspaces of $\mathbb{C}^N$).
- The condition that a line $\ell \in \mathrm{Gr}(\mathbb{C}^4, 2)$ is contained in X can translated to some algebraic equations in $\ell \in \mathrm{Gr}(\mathbb{C}^4, 2)$. Counting the number of solutions to algebraic equations is the realm of intersection theory.

Intersection theory/cohomology

Let M be a smooth and compact manifold/variety with real dimension $2n$. Its cohomology ring (with $\mathbb{Q}$ coefficients) is

$$H^*(M) = H^0(M) \oplus H^1(M) \oplus \ldots \oplus H^{2n}(M)\,.$$

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- If $Z \subseteq M$ is a closed submanifold of real codimension d, then it defines a class

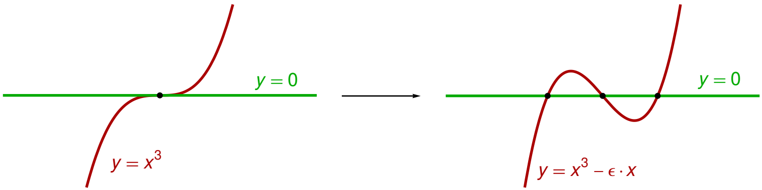
  $$[Z] \in H^d(M) .$$

  Deforming Z to Z' does not change this class $[Z] = [Z']$. Every element of $H^d(M)$ is a linear combination of such classes.
- There is a intersection product:

  $$[Z_1] \cdot [Z_2] = [Z_1 \cap Z_2] \in H^{d_1+d_2}(M)$$

  if Z_1 and Z_2 intersect nicely. If they don't, we deform one of them.

Intersection theory/cohomology



$$[Z_1] \cdot [Z_2] = 3\,[\mathrm{pt}]$$

Intersection theory/cohomology

- We have the degree functional $\int_M : H^*(M) \to \mathbb{Q}$ which is defined by
$$\int_M \alpha = \begin{cases} 0 & \text{if } \alpha \in H^{<2n}(M) \\ 1 & \text{if } \alpha = [\mathrm{pt}] \end{cases}$$
In the picture before, $\int_M [Z_1] \cdot [Z_2] = 3$.
- A rank r vector bundle V on M has Chern classes
$$c_j(V) \in H^{2j}(M) \text{ for } j = 1, 2, \ldots, r\,.$$

Back to 27

$$\#\text{lines on (generic) cubic} = \int_{\mathrm{Gr}(\mathbb{C}^4,2)} c_4(\mathrm{Sym}^3(\mathcal{V}^\vee))$$
$$= \int_{\mathrm{Gr}(\mathbb{C}^4,2)} \left(18c_1^2c_2 + 9c_2^2\right) = 18 + 9 = \boxed{27}$$

where $c_i = c_i(\mathcal{V})$ and $\mathcal{V}$ is the tautological vector bundle on $\mathrm{Gr}(\mathbb{C}^4, 2)$.

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- Calculating this kind of intersection numbers on the Grassmannian is the subject of Schubert calculus.
- $H^*(\mathrm{Gr}(\mathbb{C}^4, 2)) \simeq \mathbb{Q}[c_1, c_2]/(2c_1c_2 - c_1^3, c_2^2 - 3c_1^2c_2 + c_1^4)$.
- $c_2^2 = [\mathrm{pt}]$.

27 is nice, but...

...it's just a number. We can't really see much structure with just a number... Instead, we can

1. Study a more general class of problems, e.g. counting curves of different degree or genus (a line is a degree 1 and genus 0 curve). When we organize these numbers in a generating series, often we get interesting things like rational functions, modular forms, recursive structures, etc.
2. Study the full collection of intersection numbers on the Grassmannian, or more general moduli spaces. Leads to Virasoro constraints!

What moduli spaces?

1. Grassmannians, flag varieties. More generally, moduli spaces of quiver representations, which parametrize vector spaces together with linear maps.

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3. Moduli space of stable maps $\overline{\mathcal{M}}_{g,m}(X)$ that parametrizes a nodal genus g curve C with n marked points and a stable map $C \to X$. The intersection numbers on these are called Gromov–Witten invariants.

⚠ $\overline{\mathcal{M}}_{g,m}(X)$ is not smooth, so defining intersection numbers is more subtle. It requires a virtual fundamental class.

What moduli spaces?

4. Moduli spaces of stable vector bundles on a fixed curve C.
5. More generally, moduli spaces of sheaves on surfaces, 3-folds and (Calabi–Yau) 4-folds.

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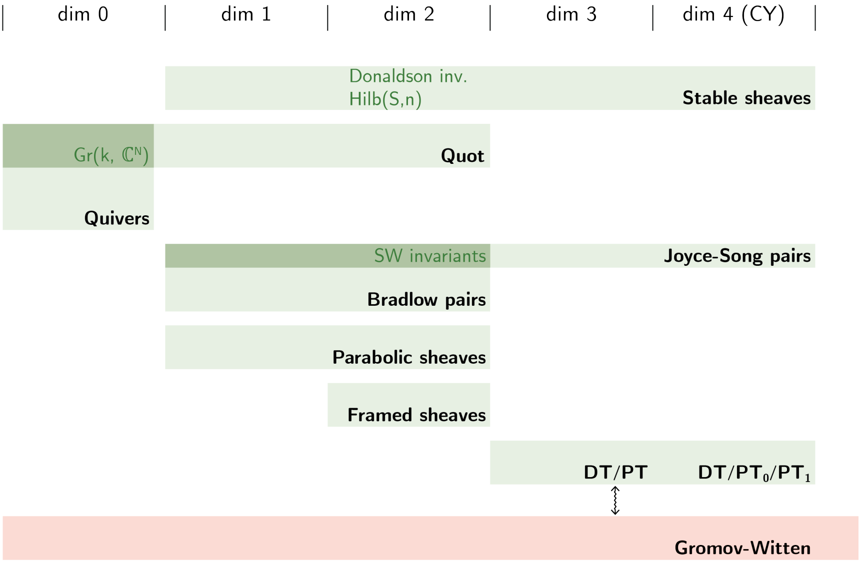
Roughly speaking, a sheaf is a singular vector bundle. We think of a sheaf F on X as a collection of vector spaces F_p over each point $p \in X$. But unlike for vector bundles, the dimension of F_p is not necessarily constant and might jump.

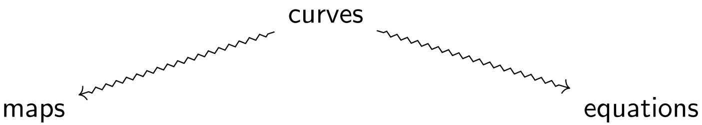
Example

- A vector bundle is a sheaf.
- If $Z \subseteq X$ is a (closed) subvariety there is a corresponding sheaf $\mathcal{O}_Z$, called the structure sheaf of Z. It has 1 dimensional fibers over $p \in Z$ and trivial fibers otherwise.

Many moduli spaces of sheaves (and related objects)



Counting curves on 3-folds



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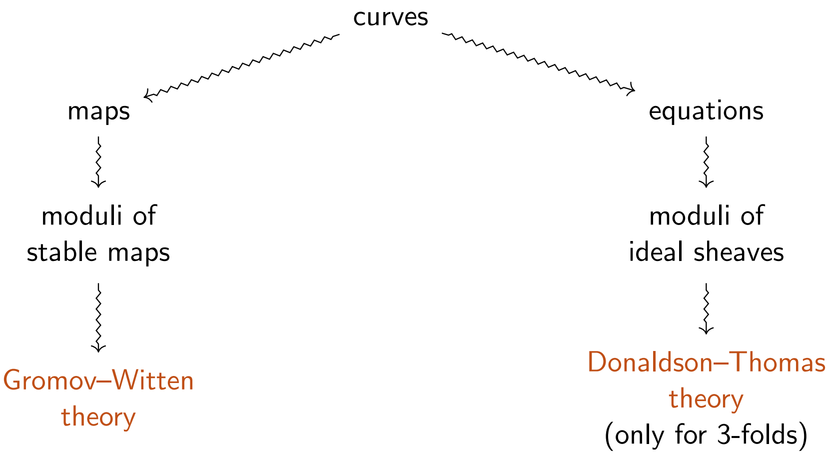

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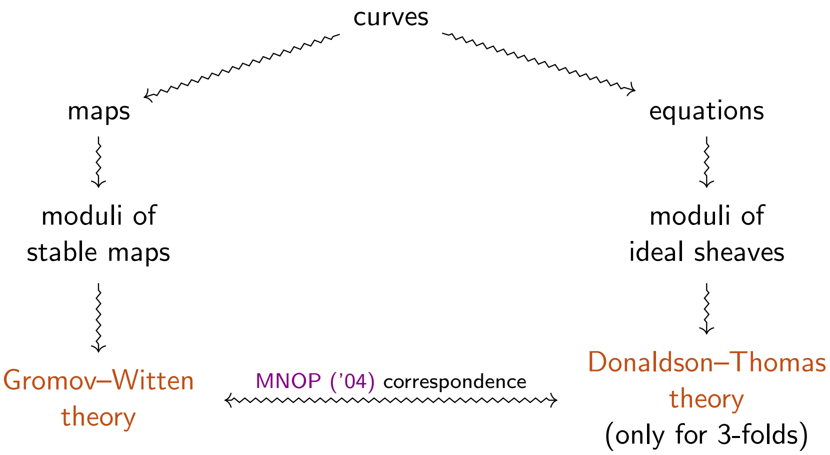

Gromov–Witten theory of the point

Even the Gromov–Witten theory of a point is highly non-trivial as it amounts to study

$$\boxed{\int_{\overline{\mathcal{M}}_{g,m}} \psi_1^{k_1} \dots, \psi_m^{k_m} \in \mathbb{Q}}$$

where $\psi_1, \dots, \psi_m \in H^2(\overline{\mathcal{M}}_{g,m})$ are certain tautological classes.

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where $\psi_1, \dots, \psi_m \in H^2(\overline{\mathcal{M}}_{g,m})$ are certain tautological classes.

We can compute these integrals thanks to a striking prediction due to Witten ('90) and proved by Kontsevich ('92).

2d gravity

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$\rightsquigarrow$

integration over
(∞-dimensional)
space of metrics on surfaces

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↓

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↙ ↘

holomorphic curves → GW of the point

triangulations of surfaces → matrix models

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GW of the point | matrix models

KdV/Virasoro constraints? $\Longleftarrow$ KdV/Virasoro constraints

2d gravity

E. Witten conjectured that the partition functions for both approaches coincide. The reason for this conjecture is an irrational (for mathematicians) idea, that gravity is unique.

M. Kontsevich '92

Witten's conjecture

Define the generating function

$$F(t_0, t_1, t_2, \ldots) = \sum_{g,m \geqslant 0} u^{2g-2} \sum_{k_1,\ldots,k_m} \frac{t_{k_1} \ldots t_{k_m}}{m!} \int_{\overline{\mathcal{M}}_{g,m}} \psi_1^{k_1} \ldots, \psi_m^{k_m}$$

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and the differential operators L_n for $n \geqslant -1$ in the variables $T_{2i+1} = t_i/(2i+1)!!$.

$$L_n = \frac{1}{4}\sum_{k+l=2n} \frac{\partial^2}{\partial T_k \partial T_l} + \frac{1}{2}\sum_{k\geqslant 0}(2k+1)T_{2k+1}\frac{\partial}{\partial T_{2k+2n+1}} - \frac{1}{2u^2}\frac{\partial}{\partial T_{2n+3}} + \frac{\delta_{n,-1}T_1^2}{4} + \frac{\delta_{n,0}}{16}$$

Witten's conjecture

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Theorem (Conjecture by Witten ('90), proof by Kontsevich ('92))

$$L_n \exp(F) = 0 \quad \text{for every } n \geqslant -1 .$$

Virasoro algebra

Definition

The Virasoro Lie algebra is the (infinite dimensional) Lie algebra Vir spanned by $\{L_n\}_{n\in\mathbb{Z}}$ and c, with Lie bracket defined by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m=0}\cdot c\,.$$
$$[L_n, c] = 0$$

Let $\mathrm{Vir}_{\geqslant -1} \subseteq \mathrm{Vir}$ be the subalgebra spanned by $\{L_n\}_{n\geqslant -1}$.

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Let $\mathrm{Vir}_{\geqslant -1} \subseteq \mathrm{Vir}$ be the subalgebra spanned by $\{L_n\}_{n\geqslant -1}$.

- The differential operators L_n in the previous slide define a representation of $\mathrm{Vir}_{\geqslant -1}$.
- It is possible to also define L_n for $n < -1$ so that we get a representation of Vir, with c acting as the identity. But the negative operators do not give more constraints.

Virasoro constraints in Gromov–Witten theory

Eguchi-Hori-Xiong (97) proposed a conjecture generalizing Witten's conjecture to the Gromov–Witten theory of X.

Virasoro constraints in Gromov–Witten theory

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Known in two large families:

- When X is a curve, by work of Okounkov-Pandharipande (03).
- When X is toric, by work of Givental (01) or more generally when X is semisimple by Teleman (07) classification theorem .

Counting curves on 3-folds

curves

maps

moduli of stable maps

Gromov–Witten theory

equations

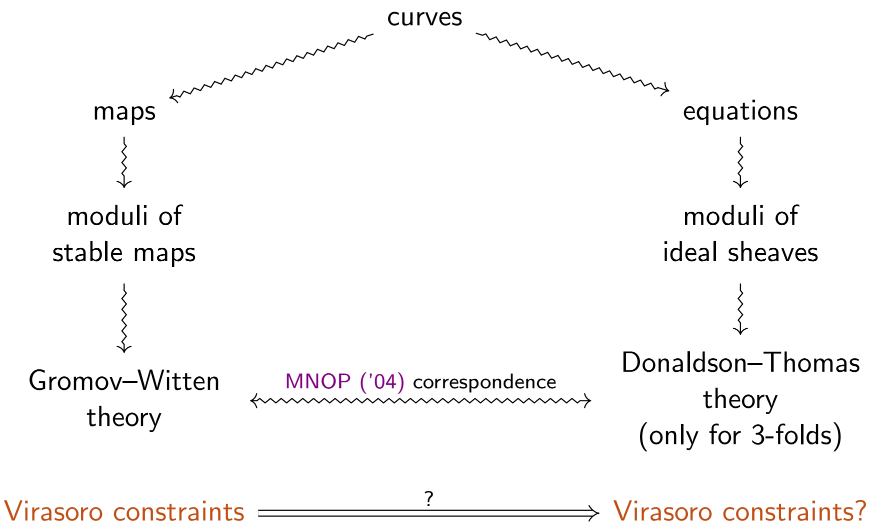
moduli of ideal sheaves

Donaldson–Thomas theory (only for 3-folds)

MNOP ('04) correspondence

Virasoro constraints

Counting curves on 3-folds



Survey paper by Pandharipande, '17

An ideal path to finding the constraints for stable pairs would be to start with the explicit Virasoro constraints in Gromov–Witten theory and then apply the correspondence. However, our knowledge of the correspondence matrix is not yet sufficient for such an application.

Another method is to look experimentally for relations which are of the expected shape. In a search conducted almost 10 years ago with A. Oblomkov and A. Okounkov, we found a set of such relations for the theory of ideal sheaves for every nonsingular projective 3-fold X. As an example, the equations for $\mathbf{P}^3$ *are presented here for stable pairs.*

R. Pandharipande, '17

Survey paper by Pandharipande, '17

Definition 3. *Let* $\mathcal{L}_k : \mathbb{D}^+ \to \mathbb{D}^+$ *for* $k \geq -1$ *be the operator*

$$\begin{aligned} \mathcal{L}_k &= -2 \sum_{a+b=k+2} (-1)^{d^L d^R} (a + d^L - 3)!(b + d^R - 3)!\, \mathsf{ch}_a \mathsf{ch}_b(\mathsf{H}) \\ &\quad + \sum_{a+b=k} a!b!\, \mathsf{ch}_a \mathsf{ch}_b(\mathsf{p}) \\ &\quad + \mathrm{R}_k + (k+1)!\, \mathrm{R}_{-1} \mathsf{ch}_{k+1}(\mathsf{p}) \,. \end{aligned}$$

Conjecture 8 (Oblomkov-Okounkov-P.). *We have*

$$\mathsf{Z}_{\mathsf{P}}(\mathsf{P}^3; q \mid \mathcal{L}_k\, \mathsf{D})_{d\mathsf{L}} = 0$$

for all $k \geq -1$*, for all* $\mathsf{D} \in \mathbb{D}^+$*, and for all curve classes* $d\mathsf{L}$.

M–Oblomkov–Okounkov–Pandharipande, '20

After the survey paper, a lot of progress was made in the "correspondence matrix".

The GW/PT transformation restricted to the essential descendents is a linear map

$$\mathfrak{C}^\bullet : \mathbb{D}^{X\star}_{\mathrm{PT}} \to \mathbb{D}^{X}_{\mathrm{GW}}$$

satisfying

$$\mathfrak{C}^\bullet(1) = 1$$

and is defined on monomials by

$$\mathfrak{C}^\bullet\Big(\widetilde{\mathrm{ch}}_{k_1}(\gamma_1)\dots\widetilde{\mathrm{ch}}_{k_m}(\gamma_m)\Big) = \sum_{P \text{ set partition of } \{1,\dots,m\}} \prod_{S\in P} \mathfrak{C}^\circ\Big(\prod_{i\in S}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\Big).$$

The operations $\mathfrak{C}^\circ$ on $\mathbb{D}^{X\star}_{\mathrm{PT}}$ are

$$\begin{aligned}\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\Big) &= \frac{1}{(k_1+1)!}\mathfrak{a}_{k_1+1}(\gamma) + \frac{(\imath u)^{-1}}{k_1!}\sum_{|\mu|=k_1-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1)}{\mathrm{Aut}(\mu)}\\ &\quad+\frac{(\imath u)^{-2}}{k_1!}\sum_{|\mu|=k_1-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{(k_1-1)!}\sum_{|\mu|=k_1-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)}, \qquad (1.14)\end{aligned}$$

$$\begin{aligned}\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\Big) &= -\frac{(\imath u)^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{(\imath u)^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma'\cdot c_1)\\ &\quad-\frac{(\imath u)^{-2}}{k_1!k_2!}\sum_{|\mu|=k_1+k_2-2}\max(\max(k_1,k_2),\max(\mu_1+1,\mu_2+1))\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\gamma'\cdot c_1), \qquad (1.15)\end{aligned}$$

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\widetilde{\mathrm{ch}}_{k_3+2}(\gamma'')\Big) = \frac{(\imath u)^{-2}|k|}{k_1!k_2!k_3!}\mathfrak{a}_{|k|-1}(\gamma\gamma'\gamma''), \quad |k| = k_1+k_2+k_3. \qquad (1.16)$$

M–Oblomkov–Okounkov–Pandharipande, ’20

And that allowed to (partially) connect the Virasoro constraints on the two sides.

Theorem 3.1. *For all* $k \geq 1$ *and* $D \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}$, *we have*

$$\mathfrak{C}^\bullet \circ \mathrm{L}_k^{\mathrm{PT}}(D) = (\imath u)^{-k}\, \widetilde{\mathrm{L}}_k^{\mathrm{GW}} \circ \mathfrak{C}^\bullet(D)$$

after the restrictions $\tau_{-2}(\mathsf{p}) = 1$ *and* $\tau_{-1}(\gamma) = 0$ *for* $\gamma \in H^{>2}(X)$.

So if we know the (stationary) Virasoro constraints on one side and we know that the correspondence holds for X, then we get the (stationary) Virasoro constraints on the other side.

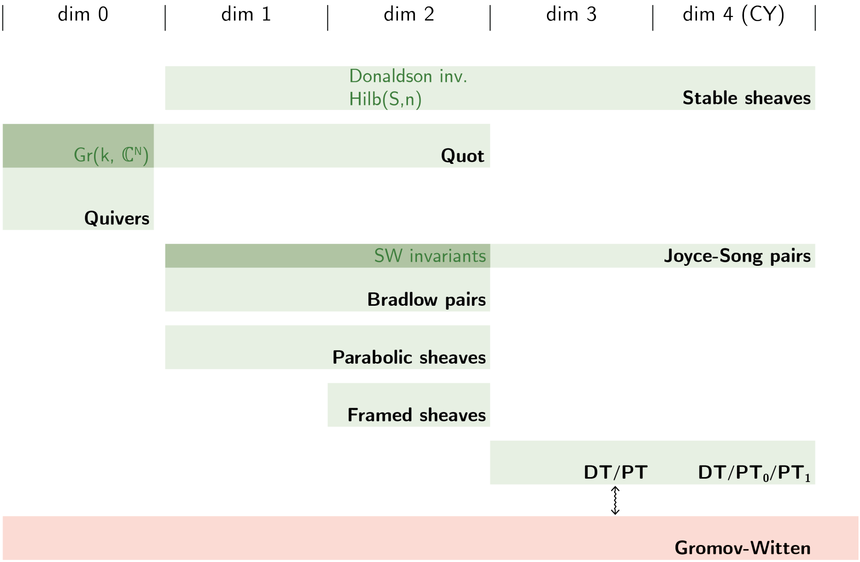
Bojko–Lim–M, 2022

But we realized soon after that Virasoro constraints are actually a much more general phenomena, not just for Donaldson–Thomas theory, but for any sheaf counting theory.

Conjecture 2.15 *Let $M = M_\alpha$ be a moduli of sheaves as in Sect. 1.2 and let $\mathbb{G}$ be a δ-normalized universal sheaf. Then*

$$\int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(\mathsf{L}_k^{\delta}(D)) = 0 \quad \textit{for any } k \geq -1,\ D \in \mathbb{D}^X.$$

Many moduli spaces of sheaves (and related objects)



Virasoro constraints for the Grassmannian

In our calculation leading to 27, we looked at intersection numbers of Chern classes $c_i = c_i(\mathcal{V})$. But the Virasoro constraints are most naturally written using

$$p_i = i!\mathrm{ch}_i(\mathcal{V})\,.$$

p_i are to c_i as power sum symmetric polynomials are to elementary symmetric polynomials.

Virasoro constraints for the Grassmannian

We consider the Grassmannian $\mathrm{Gr}(\mathbb{C}^N, k)$.

Definition

Let $\mathbb{D} = \mathbb{Q}[p_1, p_2, p_3, \ldots]$ be the Grassmannian descendent algebra. Define Virasoro operators $L_n \colon \mathbb{D} \to \mathbb{D}$ for $n \geqslant -1$ by

$$\mathsf{L}_n = \sum_{j \geqslant 0} j p_{n+j} \frac{\partial}{\partial p_j} + \sum_{a+b=n} p_a p_b + (2k - N) p_n \,.$$

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Theorem (Bojko–Lim–Moreira, '23)

For every $n > 0$ and $D \in \mathbb{D}$ we have

$$\int_{\mathrm{Gr}(\mathbb{C}^N, k)} \mathrm{L}_n(D) = 0 \,.$$

Virasoro constraints for the Grassmannian

Example

$$\int_{\mathrm{Gr}(\mathbb{C}^4,2)} \mathsf{L}_1(p_1^3) = 3\int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_1^2 p_2 = 0$$

$$\int_{\mathrm{Gr}(\mathbb{C}^4,2)} \mathsf{L}_2(p_1^2) = 2\int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_1 p_3 + \int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_1^4 = -2+2=0$$

$$\int_{\mathrm{Gr}(\mathbb{C}^4,2)} \mathsf{L}_1(p_1 p_2) = \int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_2^2 + 2\int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_1 p_3 = 2-2=0$$

$$\int_{\mathrm{Gr}(\mathbb{C}^4,2)} \mathsf{L}_3(p_1) = \int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_4 + 2\int_{\mathrm{Gr}(\mathbb{C}^4,2)} p_1^2 p_2 = 0+0=0$$

$$p_1 = c_1, p_2 = c_1^2 - 2c_2, p_3 = c_1^3 - 3c_1 c_2, p_4 = c_1^4 - 4c_1\,.$$

Some final remarks

- We have proofs of this result for moduli of sheaves on X up to $\dim X = 2$ (with some restrictions in the $\dim X = 2$ case).
- The main tool in these proofs is wall-crossing, which sometimes allows us to reduce the constraints on a complicated space to a simpler space.
- The Virasoro constraints and wall-crossing are connected by a beautiful vertex algebra constructed by Joyce ('18). This was understood in Bojko–Lim–M ('22).

Some final remarks

- The sheaf Virasoro constraints are arguably simpler and better understood than the Gromov–Witten constraints, despite being much more recent (simpler formulas, richer class of examples/toy models, constraints on only 1 space, vertex algebra formalism, proofs for surfaces).
- Dream: prove the Gromov–Witten constraints for 3-folds by proving first the DT version and applying the correspondence.

Thank you!