The cohomology ring is not χ-independent

Def: S projective smooth surface $/\mathbb{C}$, H polarization.

Define the slope of a 1-dimensional sheaf F on S by $\mu_H(F) = \dfrac{\chi(F)}{c_1(F)\cdot H}$.

We say F is H-semistable (stable) if

$\mu_H(G) \leq \mu_H(F)$ ($<$) for any $G \subseteq F$ ($G \subsetneq F$)

Let $M_{d,\chi} := M^H_{d,\chi}(\mathbb{P}^2)$

$= \{$ semistable sheaves on $\mathbb{P}^2$ with $\deg = c_1(F)\cdot H = d$, $\chi(F) = \chi \} / S\text{-equiv}$

① $M_{d,\chi}$ is projective and has dimension d^2+1.

② If $\gcd(d,\chi) = 1$ then stable = semistable and $M_{d,\chi}$ is smooth.

③ There are basic isomorphisms

$$M_{d,\chi} \longrightarrow M_{d,\chi+d} \qquad\qquad M_{d,\chi} \longrightarrow M_{d,-\chi}$$

$$F \longmapsto F \otimes \mathcal{O}(1) \qquad\qquad F \longmapsto \mathcal{E}xt^1(F, K_{\mathbb{P}^2})$$

(4) There is a map

$$f : M_{d,\chi} \longrightarrow |\mathcal{O}(d)| := \mathbb{P} H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \cong \mathbb{P}^{\frac{d(d+3)}{2}}$$

$$F \longmapsto \operatorname{Supp} F$$

(Supp F: fitting support.)

If $[C] \in |\mathcal{O}(d)|$ is smooth curve the fiber $f^{-1}([C])$ is $\operatorname{Jac}(C)$: abelian variety of dimension $g = \frac{(d-1)(d-2)}{2}$.

E.g. if F is pushforward of vector bundle from curve C then supp F is thickening of C

Examples: If $d = 1, 2$ the fitting map is an iso

$$f : M_{1,1} \xrightarrow{\sim} \mathbb{P}^2$$

$$M_{2,1} \xrightarrow{\sim} \mathbb{P}^5$$

If $d = 3$ the fitting map is universal cubic

$$\mathcal{C} \cong M_{3,1} \longrightarrow |\mathcal{O}(3)| = \mathbb{P}^9$$

$$\cap$$

$$\mathbb{P}^2 \times |\mathcal{O}(3)|$$

I - Enumerative geometry of $K_{\mathbb{P}^2}$ & χ-independence

Let

$$\Omega_q(X) = \sum_j \dim IH^{2j}(X)\, q^{j - \dim_{\mathbb{C}} X}$$

$\dim IH^{2j}(X) = \dim H^{2j}(X)$ if X is smooth

$$\Omega^{Perv}_{(q,t)}(M_{d,\chi}) \xrightarrow{q=t} \Omega_q(M_{d,\chi}) \xrightarrow{q=1} \chi(M_{d,\chi})$$

$\Omega^{Perv}_{(q,t)}(M_{d,\chi}) \longleftrightarrow$ Refined BPS of $K_{\mathbb{P}^2}$ $N_d^{(j_L, j_R)}$

$\chi(M_{d,\chi}) = m_{g=0,d}$

Refined BPS of $K_{\mathbb{P}^2}$ $N_d^{(j_L, j_R)}$ $\rightsquigarrow$ GV of $K_{\mathbb{P}^2}$ $m_{g,d}$ $\xrightarrow{g=0}$ $m_{g=0,d}$

$N_d^{(j_L, j_R)} \longleftrightarrow$ Refined PT of $K_{\mathbb{P}^2}$

$m_{g,d} \longleftrightarrow PT(K_{\mathbb{P}^2})$

Refined PT of $K_{\mathbb{P}^2}$ $\rightsquigarrow$ $PT(K_{\mathbb{P}^2})$

$PT(K_{\mathbb{P}^2}) \overset{MNOP}{\longleftrightarrow} GW(K_{\mathbb{P}^2})$

$GW(K_{\mathbb{P}^2}) \longleftrightarrow GW(\mathbb{P}^2/E)$

Conjecture: There are non-negative integers $N_d^{(j_L, j_R)}$ for $j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ s.t.

Refined BPS

$$\Omega_q(M_{d,\chi}) = \sum_{j_L, j_R} (-1)^{2j_L + 2j_R} N_d^{(j_L, j_R)} \Omega_q(\mathbb{P}^{2j_L}) \Omega_q(\mathbb{P}^{2j_R})$$

$\Omega^{perv}_{(q,t)}$ $\quad$ $\Omega_t(\mathbb{P}^{2j_R})$

In particular, $\Omega_q(M_{d,\chi})$ should not depend on χ!

Theorem (Maulik–Shen): $\Omega_q(M_{d,\chi})$ does not depend on χ

$\Omega^{perv}_{(q,t)}(M_{d,\chi})$

(Case $\gcd(d,\chi)=1$ was known before)

On the opposite direction:

Theorem (Woolf): $M_{d,\chi} \cong M_{d,\chi'}$ as algebraic varieties iff $\chi \equiv \pm\chi' \mod d$.

Proof goes by identifying nef cone of $M_{d,\chi}$ and using the contracting morphisms associated to its rays.

Natural question: Are $M_{d,\chi}$ diffeomorphic for different χ? ($\gcd(d,\chi) = 1$)
Homeomorphic? Is $H^*(M_{d,\chi})$ also χ-independent as a ring?

No!

Theorem (Lim - M - Pi): Suppose $\gcd(d,\chi) = \gcd(d,\chi') = 1$

$H^*(M_{d,\chi}; \mathbb{C}) \cong H^*(M_{d,\chi'}; \mathbb{C})$ as graded $\mathbb{C}$-algebras

if and only if $\chi \equiv \pm \chi' \mod d$.

In particular $M_{d,\chi}$ and $M_{d,\chi'}$ are not homeomorphic.

Remark: This contrasts with the situation for CY.

i.e. if we replace $\mathbb{P}^2$ by $K3$ or T^*C (Higgs bundle)

($K3$:) Spaces are diffeo for different χ

(T^*C:) Cohomology ring isomorphic! Expected to be diffeomorphic?

From now on $\gcd(d,\chi)=1$

II - Generators & relations on $H^*(M_{d,\chi})$

We have universal sheaf $\mathbb{F}$

$$\begin{array}{ccc} & \mathbb{F} & \\ & | & \\ & \mathbb{P}^2 \times M_{d,\chi} & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathbb{P}^2 & & M_{d,\chi} \end{array}$$

We define cohomology classes

$$c_k(j) = \left[p_* \left(ch(\mathbb{F}) e^{\delta} q^* H^j \right) \right]_{2k} \in H^{2k}(M_{d,\chi})$$

where $\delta \in H^2(\mathbb{P}^2 \times M_{d,\chi})$ is a normalization factor defined so that $c_0(2) = 0$, $c_1(1) = 0$.

Remark: • $H^2(M_{d,\chi})$ is rank 2 generated by $c_1(0)$ (relatively ample wrt f) and $c_1(2) = f^* H$.

• Generators behave well wrt symmetries

$$H^*(M_{d,\chi}) \xrightarrow{\sim} H^*(M_{d,\chi+d})$$
$$c_k(j) \longmapsto c_k(j)$$

$$H^*(M_{d,\chi}) \xrightarrow{\sim} H^*(M_{d,-\chi})$$
$$c_k(j) \longmapsto (-1)^{k+j+1} c_k(j)$$

Let $T^1 = \mathbb{C}\langle c_1(0), c_1(2)\rangle$

$T^j = \mathbb{C}\langle c_j(0), c_j(1), c_j(2)\rangle, \quad j = 2, \dots, d-2$

$T = \bigoplus_{j=1}^{d-2} T^j$ ← graded vector space

Theorem (Pi–Shen + Yuan):

① T generates $H^*(M_{d,\chi})$ as a ring, i.e.

$\mathbb{C}[T] \twoheadrightarrow H^*(M_{d,\chi})$

② There are no relations (for $d \geq 4$) up to degree $d-1$, i.e.

$\mathbb{C}[T]^k \xrightarrow{\sim} H^{2k}(M_{d,\chi})$ for $0 \leq k \leq d-1$

③ For $d \geq 5$ there are exactly 3 relations in degree d, i.e.

$\ker(\mathbb{C}[T]^d \to H^{2d}(M_{d,\chi}))$ is 3-dimensional.

Proof: ① $\{c_k(\alpha) : k \geq 1\}$ generate $H^*(M_{d,\chi})$ by classical Beauville diagonal trick.

② Use "generalized Mumford relations" to write $c_k(\alpha)$ in terms of lower deg generators for $k \geq d-1$

③ Compare $\dim \mathbb{Q}[T]^h$ with Betti numbers of $M_{d,\chi}$ in stable range

Thm (Yuan): For $d \geq 5$, let $m = \binom{d-1}{2} + 1$

$$b_{2k}(M_{d,\chi}) = \begin{cases} b_{2k}\left((\mathbb{P}^2)^{[m]}\right), & k \leq d-2 \\ \quad '' \qquad -3, & k = d-1 \\ \quad '' \qquad -12, & k = d \end{cases}$$

How to get relations?

Mumford relations: Fact: $H^1(F) = H^2(F) = 0$ if $F \in M_{d,\chi}$ and $\chi \geq g = \frac{(d-1)(d-2)}{2}$

So $Rp_* \mathbb{F}$ is v.b of rank χ

$\Rightarrow \quad c_k(Rp_* \mathbb{F}) = 0$ for $k > \chi \geq$

Generalized Mumford relations:

Suppose $d' < d$ and $\frac{\chi}{d} < \frac{\chi'}{d'} < \frac{\chi}{d} + 3$

For any $F \in M_{d,\chi}$, $F' \in M_{d',\chi'}$

$$\mathrm{Hom}(F', F) = 0, \quad \mathrm{Ext}^2(F', F) = \mathrm{Hom}(F, F' \otimes \omega_{\mathbb{P}^2})^\vee = 0$$

So $-Rp_* R\mathcal{H}om(\mathbb{F}', \mathbb{F})$ is v.b. of rank dd'

$\rightsquigarrow c_h(-Rp_* R\mathcal{H}om(\mathbb{F}', \mathbb{F})) = 0$ for $h > dd'$ in $H^*(M_{d',\chi'} \times M_{d,\chi})$.

Remark: The 3 relations in degree d can be obtained in this way using $d' = 1$, $M_{1,\chi'} = \check{\mathbb{P}}^2$.

Question (Komonov - Pi): Are these all relations?

True for $d = 3, 4$.

True for $d = 5$ up to degree 16, but maybe not in general

$d = 3, \quad \chi = 1$

$\dim M_{3,1} = 10$ (above $k = 10$)

k	1	2	3	4	5	6	7	8	9	10
$b_{2k}(M_{3,1})$	2	3	3	3	3	3	3	3	2	1
# relations			1						1	

$d = 4, \chi = 1$ $\qquad$ $\dim M_{4,1} = 17$

k	1	2	3	4	5	6	7	...	12	13	14	...
$b_{2k}(M_{4,1})$	2	6	10	14	15	15	15	...	15	14	10	...
# relations				6	3				1		1	

$G_1 + G_2$ (arrow to 6), G_1 (arrow to 3), M (arrows to both 1s)

$d = 5$

k	...	5	6	7	8	9	...	14	15	16	17	18	...
# relations $M_{5,1}$		3	12	13					1		1	1	...
# relations $M_{5,2}$		3	12	13	2	1		1	1	1			...

M (arrows to 15 and 17), G_1 (arrow to 3), Missing relations (arrows to the circled 1s)

III – Proof of main theorem

Assume now $0 < x, x' <$

Suppose $\exists$

$$\begin{array}{ccc} \phi : H^\bullet(M_{d,x}) & \longrightarrow & H^\bullet(M_{d,x'}) \\ \uparrow & & \uparrow \\ \mathbb{C}[T] & \overset{\phi}{\dashrightarrow} & \mathbb{C}[T] \end{array}$$

Can lift to $\mathbb{C}[T]$ by freeness

$\phi(c_k(j)) \in H^{2k}(M_{d,x}) \cong \mathbb{C}[T]^k$ for $k \leq d-2$.

Goal: Show that there is no $\phi : \mathbb{C}[T] \to \mathbb{C}[T]$ preserving the 3-dim subspace of relations in $\deg = d$

Naive dimension count, $d = 5$.

$$\#\text{variables} = \sum_{j=1}^{d-2} \dim \mathrm{Hom}(T^j, \mathbb{C}[T]^j) = 2 \cdot 2 + 3 \cdot 6 + 3 \cdot 13 = 61$$

$\wedge$

$$\#\text{equations} = \dim Gr(3, \mathbb{C}[T]^d) = 3 \cdot (45 - 3) = 126$$

Let $R'_1, R'_2, R'_3 \subseteq \mathbb{C}[T]^d$ be the 3 relations generating $\ker(\mathbb{C}[T]^d \to H^{2d}(M_{d,x'}))$

Suppose there is $C = (c_{ij}) \in GL(3;\mathbb{C})$ s.t.

$$\phi(R_i) = \sum_{j=1}^{3} c_{ij} R_j$$

Step 1: Calculate the coefficients of terms

$$c_2(s)\, c_{d-2}(t) \in T^2 \otimes T^{d-2} \subseteq \mathbb{C}[T]^d$$

in R_i'

Let $M_i' = \begin{pmatrix} \text{coefficient of} \\ c_2(s)\, c_{d-2}(t) \text{ in} \\ R_i' \end{pmatrix}_{0 \le s,t \le 2} \in M_{3,3} \cong \mathrm{Hom}((T^{d-2})^{\vee}, T^2)$ $\quad$ ($M_{3,3} \cong T^{d-2} \otimes T^2$)

Example:

$$M_3^{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{2}{d-2} & 0 & \frac{12x^2 - 12xd - d^2}{8d(d-2)} \end{bmatrix}$$

Step 2: Find c_{ij}

Let A, B be matrices defined by the linear maps

$$T^{d-2} \hookrightarrow \mathbb{C}[T]^{d-2} \xrightarrow{\phi} \mathbb{C}[T]^{d-2} \twoheadrightarrow T^{d-2}$$

$$T^2 \twoheadrightarrow \cdots \twoheadrightarrow T^2$$

I.e. $\phi(c_{d-2}(s)) = \sum_{t=0}^{2} a_{st}\, c_{d-2}(t)$ + products of lower deg generators

Then $A^t M_i B = \sum_{j=1}^{3} c_{ij} M_j'$

Trick: Look at cubic curves

$$E^x = \{ [x:y:z] : \det\left(x M_1^x + y M_2^x + z M_3^x\right) \} \subseteq \mathbb{P}^2$$

$$\begin{array}{ccc} E & \overset{\sim}{\dashrightarrow} & E' \\ \cap\!\!\mid & & \cap\!\!\mid \\ \mathbb{P}^2 & \underset{\sim}{\overset{C}{\longrightarrow}} & \mathbb{P}^2 \end{array} \qquad (*)$$

Fact: E and E' are model elliptic curves with node at $[0:0:1]$ $\overset{(*)}{\Longrightarrow}$ $c_{31} = c_{32} = 0$

Can use $(*)$ to solve for c_{ij} completely

Solutions type I: $c_{12} = c_{21} = 0$, $c_{12} = \sqrt[3]{\frac{(2x'-d)(x'-d)x'}{(2x-d)(x-d)x}}$, ...

Solutions type II: $c_{11} = c_{22} = 0$, ...

Step 3: Solve for A, B. Linear system in entries of A, B^{-1}.

- Unique solution (up to scaling) for c_{ij} type I
- No solution for type II

Step 4: Look at more coefficients in R_i, namely

$$c_{d-2}(s)\, c_1(u)\, c_1(v)$$

New variables

$$\phi : T^2 \oplus \mathrm{Sym}(T^1) \to \mathbb{C}[T]^2 = T^2 \oplus \mathrm{Sym}(T^1)$$

$$T^2 \oplus \mathrm{Sym}(T^1) \cong \mathbb{C}^6 \xrightarrow{\begin{bmatrix} B & U \\ 0 & V \end{bmatrix}} \mathbb{C}^6 \cong T^2 \oplus \mathrm{Sym}(T^1)$$

Solve explicitly for U, V

$\leadsto$ relations among d, x, x'

$$(x - x')(d - x - x')\big(\text{complicated stuff}\big) = 0$$

$\text{complicated stuff} \neq 0$ by basic number theory + analysis

$$\Downarrow$$

$$x = x' \quad \text{or} \quad x' = d - x$$