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A rationality result on the GW of local Hirzebruch surfaces

(Joint work in progress with Tim Buelles)

1 Introduction

GW invariants of local surfaces

S = Projective smooth surface $/\mathbb{C}$

Its canonical bundle K_S is non-compact CY 3-fold

Moduli of stable maps to K_S is not necessarily compact, but we can still define its GW invariants

$$N_{g,\beta}^{K_S} := \int_{[\overline{M}_g(S,\beta)]^{vir}} e\left(-R^{\bullet}\pi_* f^* K_S\right) \in \mathbb{Q}$$

$$\shortparallel$$

$$N_\beta^g$$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & S \\ {\scriptstyle \pi}\downarrow & & \\ \overline{M}_g(S,\beta) & & \end{array}$$

As usual we can assemble them in a partition

f.

As usual we can assemble them in a partition function

$$F^{K_S} = \sum_{\substack{g \geq 0 \\ \beta \in H_2(S)}} N_{g,\beta}^{K_S} \, u^{2g-2} \, Q^{\beta}$$

$$Z^{K_S} = \exp\left(F^{K_S}\right)$$

Hirzebruch surfaces

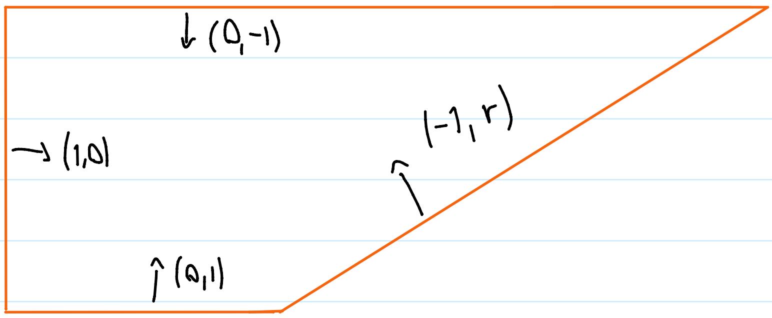
For $r \in \mathbb{Z}$ the Hirzebruch surface is defined as the projective bundle over $\mathbb{P}^1$

$$\begin{array}{c} \mathbb{F}_r = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r)\right) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

When $r = 0$ $\quad \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

$r = 1$ $\quad \mathbb{F}_1 = Bl_{pt}(\mathbb{P}^2)$

It's a toric surface with moment polytope



$H_2(S)$ is generated by F = class of fiber
and B = class of one of the sections

$$F^2 = 0 \qquad F\cdot B = 1 \qquad B^2 = -r$$
$$(B+rF)^2 = r$$

We'll denote Q_F, Q_B the respective Novikov parameters.

Statement of main result

Theorem (T. Bueller, M): Let $S = \mathbb{F}_r$ be a Hirzebruch surface.

Fix $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(S;\mathbb{Z})$ and assume

$2\beta\cdot F + g - 1 > 0$. Then

$$\sum_{i \in \mathbb{Z}} N^{g}_{\beta + iF} \, Q^{i}$$

is the Taylor expansion of a rational function of the form

$$\boxed{f(Q) = \frac{P(Q)}{(1-Q)^{4(\beta \cdot F) + 2g - 2}}}$$

P Taylor polynomial

satisfying

$$\boxed{Q^{K_S \cdot \beta} f(Q^{-1}) = f(Q)}$$

Equivalently

$$\left[Q_B^{j} u^{2g-2}\right] F^{K_S} = \sum_{i=0}^{\infty} N^{g}_{jB + iF} \, Q_F^{i}$$

is the expansion of a rational function

$$f(Q_F) = \frac{P(Q_F)}{(1-Q_F)^{4j+2g-2}} \qquad \text{where}$$

P is palindromic polynomial of degree

$$(r+2)j + 2g - 2$$

<u>Remark</u>: Case $g=0$, $r=0$ (i.e. $S = \mathbb{P}^1 \times \mathbb{P}^1$) conjectured by Klemm–Kreuzer–Riegler–Scheidegger.

More general $Q_F \leftrightarrow Q_F^{-1}$ symmetry suggested in physics literature as consequence of heterotic string + mirror symmetry.

Some examples

$r=0$, $g=0$ $(\mathbb{P}^1 \times \mathbb{P}^1)$

	0	1	2	3	4	5	6	7	8
0	0	-2	$-\frac{1}{4}$	$-\frac{2}{27}$	$-\frac{1}{32}$	$-\frac{2}{125}$	$-\frac{1}{108}$	$-\frac{2}{343}$	$-\frac{1}{256}$
1	-2	-4	-6	-8	-10	-12	-14	-16	
2	$-\frac{1}{4}$	-6	$-\frac{65}{2}$	-110	$-\frac{1155}{4}$	-644	-1281		
3	$-\frac{2}{27}$	-8	-110	$-\frac{20416}{27}$	-3556	-13072			
4	$-\frac{1}{32}$	-10	$-\frac{1155}{4}$	-3556	$-\frac{436289}{16}$				
5	$-\frac{2}{125}$	-12	-644	-13072					
6	$-\frac{1}{108}$	-14	-1281						
7	$-\frac{2}{343}$	-16							
8	$-\frac{1}{256}$								

...

$$-\frac{\frac{2}{27}(1+Q^4)+\frac{196}{27}(Q+Q^3)+\frac{100}{3}Q^2}{(1-Q)^{10}}$$

$$-\frac{\cdots 27 \cdots 27 \cdots 3 \cdots}{(1-Q)^{10}}$$

$$\sum -\frac{2}{i^3} Q^i$$

$$-\frac{2}{(1-Q)^2} \qquad -\frac{\frac{1}{4}+\frac{9Q}{2}+\frac{1}{4}Q^2}{(1-Q)^6}$$

$r=0,\ g=1$

$(P^1 \times P^1)$

	0	1	2	3	4	5	6	7	8
0	0	$-\frac{1}{6}$	$-\frac{1}{12}$	$-\frac{1}{18}$	$-\frac{1}{24}$	$-\frac{1}{30}$	$-\frac{1}{36}$	$-\frac{1}{42}$	$-\frac{1}{48}$
1	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{5}{6}$	-1	$-\frac{7}{6}$	$-\frac{4}{3}$	
2	$-\frac{1}{12}$	$-\frac{1}{2}$	$\frac{37}{6}$	$\frac{353}{6}$	$\frac{1103}{4}$	$\frac{2803}{3}$	2591		
3	$-\frac{1}{18}$	$-\frac{2}{3}$	$\frac{353}{6}$	$\frac{8576}{9}$	$\frac{22487}{3}$	$\frac{120860}{3}$			
4	$-\frac{1}{24}$	$-\frac{5}{6}$	$\frac{1103}{4}$	$\frac{22487}{3}$	$\frac{1116529}{12}$				
5	$-\frac{1}{30}$	-1	$\frac{2803}{3}$	$\frac{120860}{3}$					
6	$-\frac{1}{36}$	$-\frac{7}{6}$	2591						
7	$-\frac{1}{42}$	$-\frac{4}{3}$							
8	$-\frac{1}{48}$								

$$-\frac{\frac{1}{12}(1+Q^4)+\frac{1}{6}(Q^2+Q^3)+\frac{47}{6}Q^2}{(1-Q)^8}$$

$$\sum_{i=1}^{\infty} -\frac{Q^i}{6i} = -\frac{1}{6}\log(1-Q)$$

$$\frac{-\frac{1}{6}(1+Q^2)+\frac{1}{3}Q}{(1-Q)^4} = -\frac{1}{6}\frac{1}{(1-Q)^2}$$

$g=0,\ r=2$

$\xrightarrow{B}$

$F \downarrow$

	0	1	2	3	4	5	6	7	8
0	0	-1	$-\frac{1}{8}$	$-\frac{1}{27}$	$-\frac{1}{64}$	$-\frac{1}{125}$	$-\frac{1}{216}$	$-\frac{1}{343}$	$-\frac{1}{512}$
1	-2	-2	0	0	0	0	0	0	
2	$-\frac{1}{4}$	-4	$-\frac{1}{4}$	0	0	0	0		
3	$-\frac{2}{27}$	-6	-6	$-\frac{2}{27}$	0	0			
4	$-\frac{1}{32}$	-8	$-\frac{65}{2}$	-8	$-\frac{1}{32}$				
5	$-\frac{2}{\ldots}$	-10	-110	-110					

4	$-\frac{1}{32}$	-8	$-\frac{65}{2}$	-8	$-\frac{1}{32}$
5	$-\frac{2}{125}$	-10	-110	-110	
6	$-\frac{1}{108}$	-12	$-\frac{1155}{4}$		
7	$-\frac{2}{343}$	-14			
8	$-\frac{1}{256}$				

Same numbers that appear for $r=0$

Proposition (Konishi–Minabe): S, S' deformation equivalent nef surfaces, $\beta \in H_2(S)$ s.t. $\beta \cdot K_S < 0$

Then

$$N_{g,\beta}^{K_S} = N_{g,\beta'}^{K_{S'}}$$

- If $r \equiv r' \bmod 2$ then $\mathbb{F}_r$ and $\mathbb{F}_{r'}$ are deformation equivalent
- $\mathbb{F}_r$ is nef iff $|r| \leq 2$

$r=3, \; g=2$

	0	1	2	3	4	5	6	7	8
0	0	$\frac{1}{240}$	$\frac{1}{240}$	$\frac{1}{48}$	$-\frac{289}{48}$	$\frac{3223}{20}$	$-\frac{709589}{240}$	$\frac{364485}{8}$	$-\frac{10145271}{16}$
1	$-\frac{1}{120}$	$\frac{1}{120}$	$-\frac{1}{120}$	$\frac{1}{40}$	$-\frac{1693}{120}$	$\frac{15197}{30}$	$-\frac{462853}{40}$	$\frac{2555293}{12}$	
2	$-\frac{1}{60}$	$\frac{1}{80}$	$\frac{1}{240}$	$\frac{1}{24}$	$-\frac{667}{30}$	$\frac{113561}{120}$	$-\frac{2041203}{80}$		
3	$-\frac{1}{40}$	$\frac{1}{48}$	$-\frac{1}{60}$	$\frac{1}{12}$	$-\frac{3641}{120}$	$\frac{84983}{60}$			
4	$-\frac{1}{30}$	$\frac{7}{240}$	$-\frac{1}{40}$	$\frac{5}{48}$	$-\frac{9251}{240}$				
5	$-\frac{1}{24}$	$\frac{3}{80}$	$-\frac{1}{6}$	$\frac{21}{80}$					
6	$-\frac{1}{20}$	$\frac{11}{240}$	$-\frac{299}{24}$						
7	$-\frac{7}{120}$	$\frac{13}{240}$							
8	$-\frac{1}{15}$								

$$\frac{1}{240(1-Q)^{10}}\Big((1+Q^{12})-12(Q+Q^{11})+66(Q^2+Q^{10})$$
$$-224(Q^3+Q^9)+529(Q^4+Q^8)$$
$$-952(Q^5+Q^7)+1456Q^6\Big)$$

The case $g=0$, $S=\mathbb{P}^1\times\mathbb{P}^1$

① $$\sum_{i=0}^{\infty} N_{ij}\,Q^i = \frac{P_{2j-2}(Q)}{(1-Q)^{4j-2}}$$

② $$N_{ij}=N_{ji}$$

③ $$N_{i0}=-\frac{2}{i^3}$$

completely determine $F_{g=0}^{K_{\mathbb{P}^1\times\mathbb{P}^1}}=\sum_{i,j\geq 0} N_{ij}\,Q_1^i\,Q_2^j$

Question: Can we give closed form expression?

Proposition:

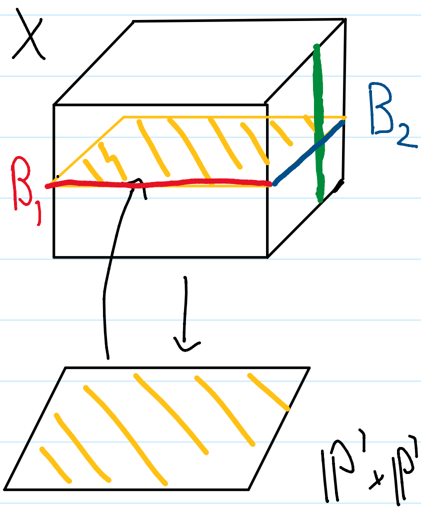
$$F_{g=0}^{K_{\mathbb{P}^1\times\mathbb{P}^1}}(Q,-1)=\frac{3}{2}\zeta(3)+\frac{1}{2}\mathrm{Li}_3(-Q)$$

$$F^{K_{\mathbb{P}^1\times\mathbb{P}^1}}_{g=0}(-1,-1) = 2\zeta(3)$$

Generalizations for CY-3

Our original motivation comes from physicists insight onto STU model

$$\begin{tikzcd} E \arrow[r] & X \arrow[d] \\ & \mathbb{P}^1\times\mathbb{P}^1 \arrow[u, bend left] \end{tikzcd}$$



$$\beta(i,j,h) = i B_1 + j B_2 + h E$$

KKRS suggest some relation

$$N^g_{(i,j,h)} \sim N^g_{(h-i-2j,\, j,\, h)}$$

Holds exactly when $j=0$ by monodromy of K3

When $h=0$ we're in local $\mathbb{P}^1 \times \mathbb{P}^1 \curvearrowleft$ own f.e.

Idea: Work in moduli of stable pairs and follow strategy of the $q \leftrightarrow q^{-1}$ f.e (Bridgeland) with derived equivalence + wall-crossing.

More generally we can consider elliptic fibrations over Hirzebruch

$$\begin{tikzcd} E \arrow[r] & X \arrow[d] & CY\text{-}3 \\ & \mathbb{F}_r \arrow[u, bend left] & \end{tikzcd}$$

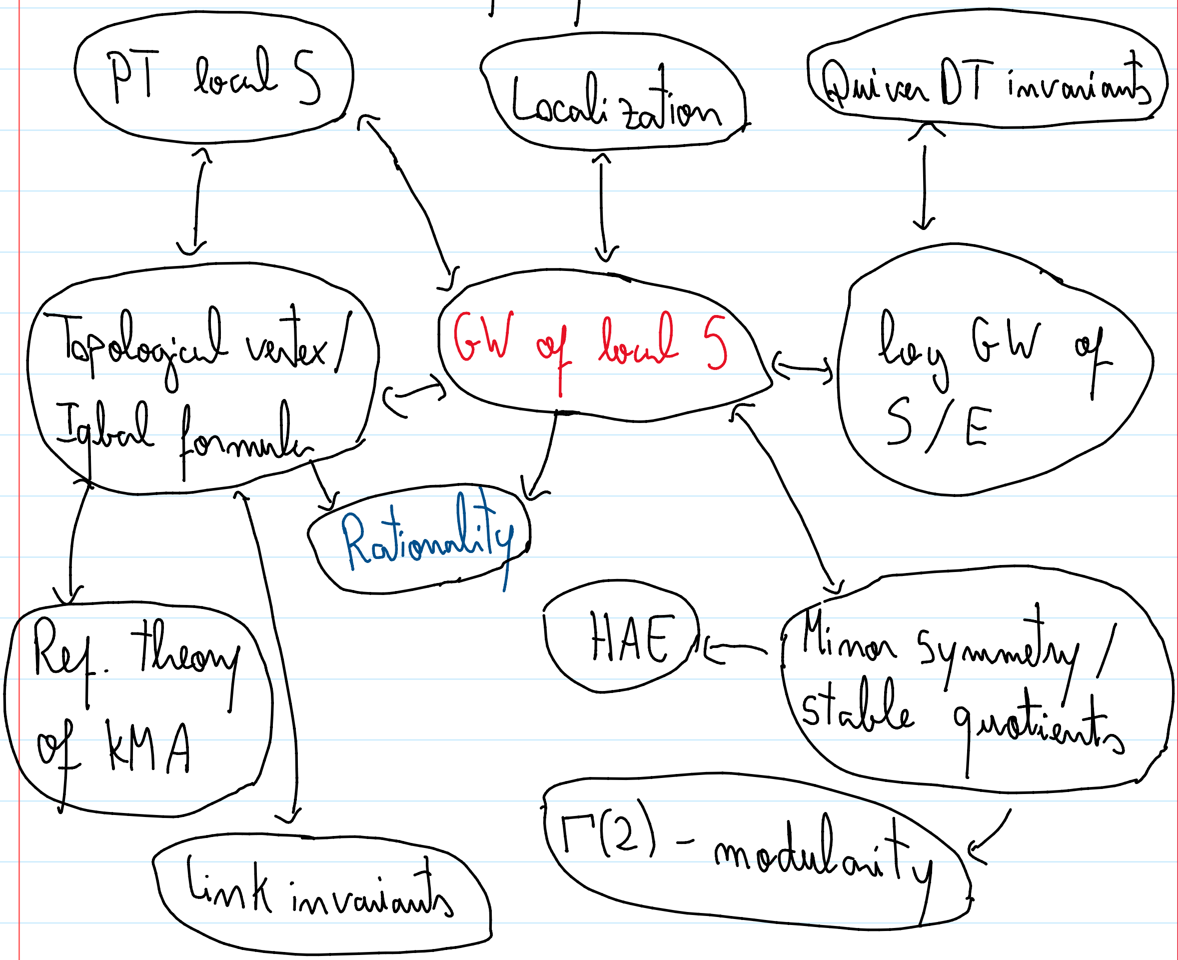
(Katz - Klemm - Vafa)

Question: What's the general geometric setup where these symmetries are expected?

Question (Probably naive): Is every CY-3 with a K3-fibration over $\mathbb{P}^1$ deformation equivalent to an elliptic fibration over a Hirzebruch (with section)?

Related subjects

(far from exhaustive)



② Iqbal's formula / Topological vertex

Very compact way to write the GW invariants of local toric surface/ toric 3-fold.

Essentially GW / PT + PT localization

Schur polynomials

Given partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_{\ell(\mu)})$ and

and a set of ordered variables $x = (x_1, x_2, \dots)$

define

$$\boxed{s_\mu(x) = \sum_{T \in SSYT(\mu)} x^T}$$

Sum over semi-standard Young Tableux

Example with $\mu = (4, 2, 1)$

Example with $\mu = (4, 2, 1)$

$$\begin{array}{|c|c|c|c|} \hline x_4 \\ \cline{1-2} x_3 & x_4 \\ \hline x_1 & x_1 & x_2 & x_2 \\ \hline \end{array} \rightsquigarrow x_1^2 x_2^2 x_3 x_4^2$$

strictly increasing

weakly increasing

Schur polynomials admit a lot of nice representations.

Weyl formula:

$$S_\mu(x_1, \dots, x_m) = \left(\prod_{i<j} (x_i - x_j) \right)^{-1} \begin{vmatrix} x_1^{\mu_1+m-1} & \cdots & x_m^{\mu_1+m-1} \\ x_1^{\mu_2+m-2} & \cdots & x_m^{\mu_2+m-2} \\ \vdots & & \\ x_1^{\mu_m} & \cdots & x_m^{\mu_m} \end{vmatrix}$$

Jacobi-Trudi formula:

$$S_\mu(x) = \det \left[h_{\mu_i + j - i}(x) \right]_{1 \le i,j \le m}$$

complete homogeneous symmetric polynomials

If we consider the specialization $x = (1, q, q^2, \dots)$

infinite set of variables

infinite set of variables

$$h_k(1,q,\dots) = \prod_{j=1}^{k} \frac{1}{1-q^j}.$$

Hook-content formula

$$s_\mu(1,q,q^2,\dots) = q^{n(\mu)} \prod_{\square\in\mu} \frac{1}{1-q^{h(\square)}}$$

$n(\mu) = \sum (i-1)\mu_i$ $\qquad$ $h(\square)$ = hook-length

Schur polynomials admit several generalizations. E.g. skew Schur polynomials $s_{\mu/\lambda}(x)$.

$s_{\mu/\lambda} = 0$ unless $\lambda \subseteq \mu$ in which case it's defined as sum over SSYT supported on skew-shape μ/λ.

The functions W_μ, $W_{\mu\nu}$

The 1-leg and 2-leg cases of the topological

The 1 leg and 2 leg cases of the topological vertex have simple combinatorial expressions in terms of Schur polynomials.

Write

$$\boxed{K(\mu) = \sum_{i=1}^{\ell(\mu)} \mu_i (\mu_i - 2i + 1)}$$

We define

$$\boxed{W_\mu(q) = (-1)^{|\mu|} q^{K(\mu)/2 + |\mu|/2} \, s_\mu(1, q, q^2, \ldots)}$$

Equivalently

$$W_\mu(q) = \frac{q^{K(\mu)/4}}{\prod_{\square \in \mu} \left(q^{h(\square)/2} - q^{-h(\square)/2} \right)}$$

Now the 2-leg version is defined as

$$\boxed{W_{\mu\nu}(q) = q^{|\nu|/2} W_\mu(q) \, s_\nu(q^{\mu_1 - 1}, q^{\mu_2 - 2}, \ldots)}$$

Proposition (J. Zhou):

$$W_{\mu\nu} = W_{\nu\mu}$$

Proof:

$$W_{\mu\nu} = (-1)^{|\mu|+|\nu|} \, q^{\frac{\kappa(\mu)+\kappa(\nu)+|\mu|+|\nu|}{2}} \sum_{\lambda} q^{-|\lambda|} s_{\mu/\lambda} \, s_{\nu/\lambda} \, . \square$$

$$\begin{aligned}
\mathcal{W}_{\underbrace{\Box\Box\Box\cdots\Box}_{n}} &= \frac{q^{n(n-1)/4}}{\prod_{k=1}^{n}(q^{k/2}-q^{-k/2})}, \\
\mathcal{W}_{A_n} &= \frac{q^{-n(n-1)/4}}{\prod_{k=1}^{n}(q^{k/2}-q^{-k/2})}, \\
\mathcal{W}_{\Box,\Box} &= \frac{((q+q^{-1}-1)}{(q^{1/2}-q^{-1/2})^2}) \\
\mathcal{W}_{\Box\Box,\Box} &= \frac{1}{(q^{1/2}-q^{-1/2})^3}\frac{(1-q^2+q^3)}{1+q} \\
\mathcal{W}_{\substack{\Box\\\Box},\Box} &= \frac{1}{(q^{1/2}-q^{-1/2})^3}\frac{q^{-2}(1-q+q^3)}{1+q} \\
\mathcal{W}_{\Box\Box\Box,\Box} &= \frac{1}{(q^{1/2}-q^{-1/2})^4}\frac{q^2(1-q^3+q^4)}{(1+q)(1+q+q^2)} \\
\mathcal{W}_{\substack{\Box\Box\\\Box},\Box} &= \frac{1}{(q^{1/2}-q^{-1/2})^4}\frac{1-q+q^2-q^3+q^4}{1+q+q^2},
\end{aligned}$$

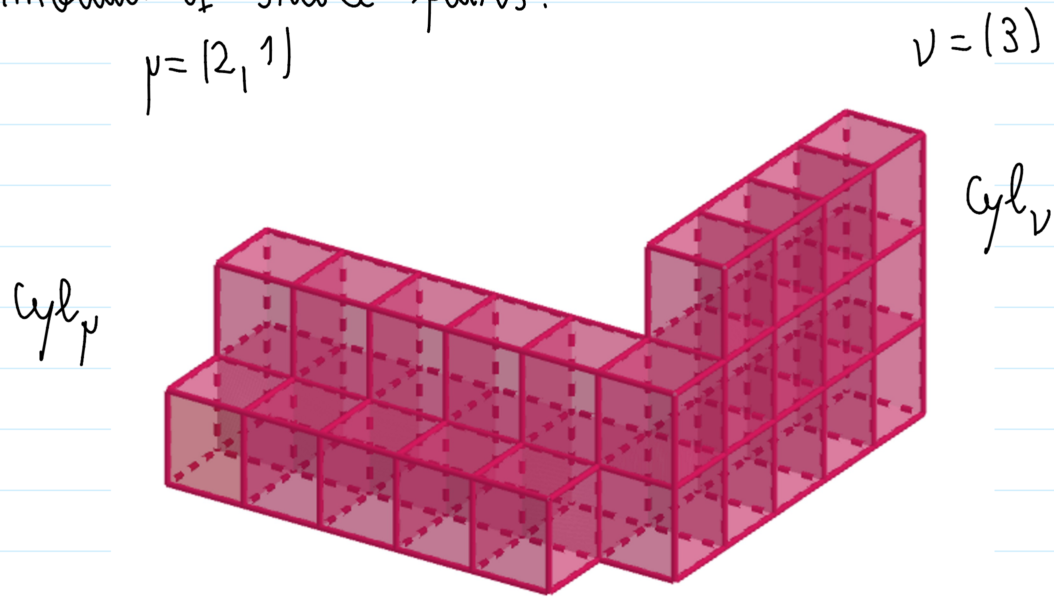
(from paper by A. Iqbal)

Remark: These $W_{\mu\nu}$ are related to several topics

- Representation theory of certain KMA (U(N) WZW theory)
- Link invariants

• Chern-Simons theory of manifold with genus 1 boundary.

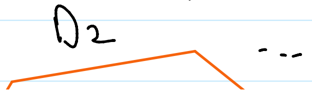
Remark: $W_{\mu\nu}$ also admits a combinatorial description motivated by localization on the moduli of stable pairs.

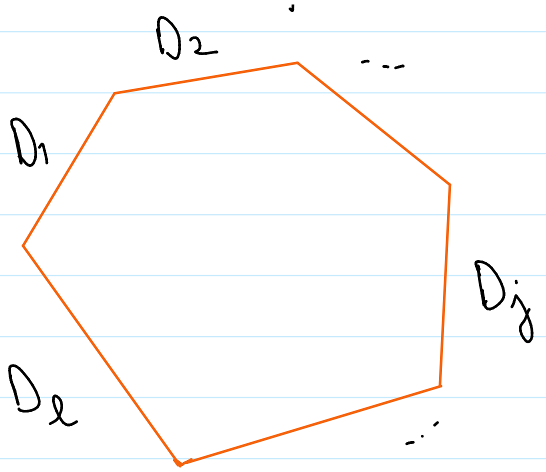


$W_{\mu\nu}$ = gen. function of box configurations supported in such regions w/ $(1,1,1)$ gravity.

Iqbal's formula

S toric surface. Moment polytope:





$S_j = D_j^2$

$Q^{D_j} \in$ Novikov ring

Theorem (J. Zhou): We have

$$\boxed{Z^{K_S} = \sum_{\mu_1,\dots,\mu_\ell} \prod_{j=1}^{\ell} \left(q^{\kappa(\mu_j) S_j/2} \times (-1)^{S_j |\mu_j|} \times W_{\mu_j,\mu_{j+1}}(q) \times Q^{|\mu_j| D_j} \right)}$$

after the change of variables $\boxed{q = e^{iu}}$

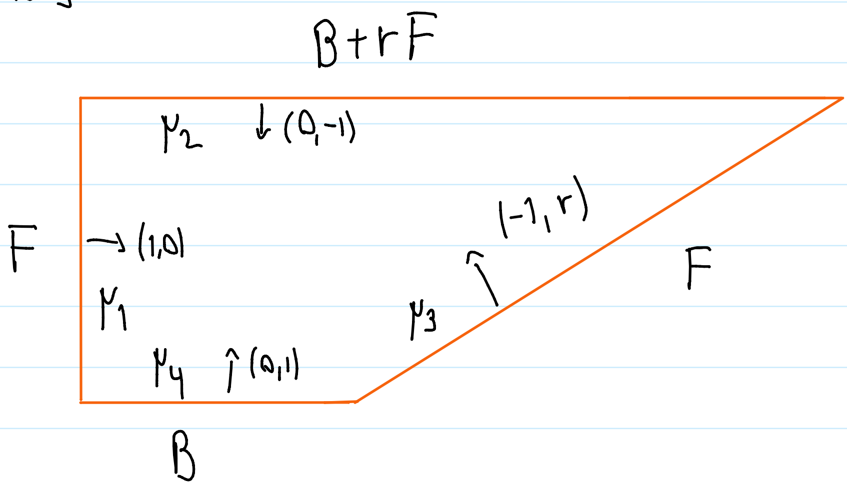
- Conjectured by A. Iqbal (2002) in several cases ($S = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, $Bl_p(\mathbb{P}^1 \times \mathbb{P}^1)$, etc.)

- Proven by J. Zhou (2003) using a formula for Hodge integrals from C-C Liu – K. Liu – J. Zhou

- Equivalent to PT/GW correspondence for local toric surfaces.

③ Proof of main theorem

For the Hirzebruch surface Iqbal's formula becomes



$$Z^{K_{F_r}} = \sum_{\mu_j} \Big(q^{r(\kappa(\mu_2)-\kappa(\mu_4))} W_{\mu_1\mu_2} W_{\mu_2\mu_3} W_{\mu_3\mu_4} W_{\mu_4\mu_1} \big((-1)^r Q_B\big)^{|\mu_2|+|\mu_4|} Q_F^{\,|\mu_1|+|\mu_3|+r|\mu_2|} \Big)$$

$$\sum^{\infty} \ldots \sum \Big(q^{r(\kappa(\mu_2)-\kappa(\mu_4))} \ldots Q^{r|\mu_2|}$$

$$= \sum_{j=0}^{\infty} Q_B^j (-1)^{rj} \sum_{|\mu_2|+|\mu_4|=j} \left(q^{r(\kappa(\mu_2)-\kappa(\mu_4))} Q_F^{r|\mu_2|} \right.$$

$$\left. \times \left(\sum_{\mu} W_{\mu\mu_2} W_{\mu\mu_4} Q_F^{|\mu|} \right)^2 \right)$$

$$\log(S_{\emptyset\emptyset}) = \sum_{k=1}^{\infty} \frac{(qQ_F)^k}{k(1-q^k)^2}.$$

$$\log(S_{\{1\}\emptyset}) = \log\left(\frac{q^{1/2}}{(1-q)^2}\right) + \sum_{k=1}^{\infty} \frac{Q_F^k(1-q^k+q^{2k})}{k(1-q^k)^2}.$$

$$\log(S_{\{1\}\{1\}}) = \log\left(\frac{q}{(1-q)^2}\right) + \sum_{k=1}^{\infty} \frac{Q_F^k(1-q^k+q^{2k})^2}{kq^k(1-q^k)^2}.$$

$$\log(S_{\{2\}\{1\}}) = \log\left(\frac{q^{5/2}}{(1-q)^3(1+q)}\right) + \sum_{k=1}^{\infty} \frac{Q_F^k(1-q^k+q^{2k})(1-q^{2k}+q^{3k})}{kq^k(1-q^k)^2}.$$

Proposition (Eguchi–Kanno): Let

$$P_\mu(q) = \sum_{i=1}^{\infty} q^{\mu_i - i} = \frac{q^{-\ell(\mu)}}{q-1} + \sum_{i=1}^{\ell(\mu)} q^{\mu_i - i}$$

Then

$$\boxed{S_{\mu\nu} = W_\mu W_\nu \exp\left(\sum_{k=1}^{\infty} P_\mu(q^k) P_\nu(q^k) \frac{(qQ_F)^k}{k} \right)}$$

Proof: Use Cauchy identity

$$\sum_{\mu} s_\mu(x)\, s_\mu(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} . \square$$

Formula for $S_{\mu\nu}$ interacts very well w/ $q = e^{iu}$!

Write the u-expansion

$$\boxed{\begin{aligned} \rho_\mu(e^{iu})\, \rho_\nu(e^{iu})\, e^{iu} &= \sum_{n=-2}^{\infty} c_n^{\mu\nu} u^n \\ &= -\frac{1}{u^2} + \left(|\mu| + |\nu| - \frac{1}{12} \right) + \frac{iu}{2} \left(\kappa(\mu) + \kappa(\nu) \right) + \ldots \end{aligned}}$$

So

$$S_{\mu\nu} = W_\mu W_\nu \exp\left(\sum_{n=-2}^{\infty} c_n^{\mu\nu} u^n \, \mathrm{Li}_{1-n}(Q_F) \right)$$

Where $\mathrm{Li}_s(Q) = \sum_{k=1}^{\infty} k^{-s} Q^k$

Example:

$$\mathrm{Li}_1(Q) = -\log(1-Q) \qquad \mathrm{Li}_{-1}(Q) = \frac{Q}{(1-Q)^2}$$

$$\boxed{\mathrm{Li}_0(Q) = \frac{Q}{1-Q}} \qquad \mathrm{Li}_{-2}(Q) = \frac{Q(1+Q)}{(1-Q)^3}$$

Fact: For $n > 1$, $\mathrm{Li}_{1-n}(Q)$ is rational function with denominator $(1-Q)^n$ and satisfies

$$\boxed{\mathrm{Li}_{1-n}(Q) = (-1)^n \,\mathrm{Li}_{1-n}(Q^{-1}).}$$

Let

$$\tilde{F} = F + \frac{2}{u^2}\mathrm{Li}_3(Q_F) + \frac{1}{6}\mathrm{Li}_1(Q_F)$$

$$\tilde{Z} = \exp(\tilde{F})$$

Algebraic manipulation shows

$$[Q_B^j]\,\tilde{Z} = \sum_{|\mu_2|+|\mu_4|=j} q^{r(\kappa(\mu_2)-\kappa(\mu_4))}\, Q_F^{r|\mu_2|}\,(1-Q_F)^{-2j}$$

$$[\Psi_B] = \sum_{|\mu_2|+|\mu_4|=j} \quad \cdots \quad Q_F \quad (1 - Q_F)$$

$$\cdot \left(\widetilde{W}_{\mu_2} \widetilde{W}_{\mu_4} \exp\left(\frac{iu}{4} \left(\kappa(\mu_2) + \kappa(\mu_4) \right) \frac{1+Q_F}{1-Q_F} \right) \right.$$

$$\left. \cdot \prod_{m=2}^{\infty} \exp\left(u^m c_m^{\mu\nu} \, \mathrm{Li}_{1-m}(Q_F) \right) \right)^2$$

Define: $C_{a,b} \subseteq \mathbb{C}[u^{-1}, u]][[Q_F]]$ as set

$$\left\{ \sum f_k(Q_F) u^k : \ f_k(Q_F) = \frac{p_k(Q_F)}{(1-Q_F)^{k+a}} \right.$$

$$\left. \text{and } Q_F^b f_k(Q_F^{-1}) = f_k(Q_F) \right\}$$

- $C_{0,b} \subseteq C_{1,b} \subseteq C_{2,b} \subseteq \ldots$
- $C_{a_1,b_1} \cdot C_{a_2,b_2} \subseteq C_{a_1+a_2, b_1+b_2}$
- $u^{-k} \in C_{-k,0}$
- $u^m \mathrm{Li}_{1-m}(Q_F) \in \widetilde{C}_{0,0}$ for $m \geq 2$

To finish pair terms with (μ_2,μ_4) & (μ_4,μ_2)

$$q^{r(\kappa(\mu_2)-\kappa(\mu_4))}\, Q_F^{r|\mu_2|} + q^{r(\kappa(\mu_4)-\kappa(\mu_2))}\, Q_F^{r|\mu_4|}$$

$$\in \widetilde{C}_{0,rj}$$

We conclude

$$[Q_B^j]\,\widetilde{Z} \in \widetilde{C}_{4j,\, j(r+2)}$$

$$\Downarrow$$

$$[Q_B^j]\,\widetilde{Z} \in C_{4j,\, j(r+2)}$$

only even powers of u.

$\square$

<u>Corollary</u>: For fixed g, j, r

$$\left| N^{K_{\mathbb{F}_r}}_{g,(1,j)} \right| \sim \frac{\gamma_{g,j}}{(4j+2g-3)!}\, r^{4j+2g-3}$$

where $\gamma_{g,j}$ is a constant not depending on r.

Conjectured (for $g=0$) by KKV

"field theory limit"

$\gamma_{0,1} = 2$	$\gamma_{1,1} = 0$	$\gamma_{2,1} = 0$
$\gamma_{0,2} = 5$	$\gamma_{1,2} = 8$	$\gamma_{2,2} = 11$
$\gamma_{0,3} = 48$	$\gamma_{1,3} = \frac{1024}{3}$	$\gamma_{2,3} = 1872$
$\gamma_{0,4} = \frac{1469}{2}$	$\gamma_{1,4} = 13176$	$\gamma_{2,4} = 171201$

④ Mirror symmetry

We now focus on $S = \mathbb{P}^1 \times \mathbb{P}^1$

The mirror of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the following family of elliptic curves:

$$\Sigma = \{ x_0^2 y_0^2 + z_1 x_1^2 y_0^2 + x_0^2 y_1^2 + z_2 x_1^2 y_1^2 + x_0 x_1 y_0 y_1 = 0 \}$$

$$\cap\!|$$

$$\mathbb{P}^1_{[x_0:x_1]} \times \mathbb{P}^1_{[y_0:y_1]}$$

z_1, z_2 are parameters related to the Novikov parameters by mirror map

$$Q_i = z_i \exp\left(\sum_{k,l \geq 0} \frac{(2k+2l-1)!}{k!^2 \, l!^2} z_1^k z_2^l \right)$$

$\Gamma(2)$ Quasimodularity:

(from Aganagic - Bouchard - Klemm)

For each z_1, z_2 the j-invariant of the elliptic curve $\Sigma_{z_1 z_2}$ in the mirror family is:

$$j(\tau) = \frac{64(3+u^2)^3}{(u^2-1)^2}$$

with

$$u = \frac{1}{8\sqrt{z_1 z_2}} - \frac{1}{2}\left(\sqrt{\frac{z_1}{z_2}} + \sqrt{\frac{z_2}{z_1}}\right)$$

ABK show: Mirror family Σ has $\Gamma(2)$ monodromy

$$\frac{z_1}{z_2} = \frac{Q_1}{Q_2}$$

$$F_g^{K_{\mathbb{P}^1 \times \mathbb{P}^1}}(Q_1, Q_2) = F_g(q_m, \tau)$$

is $\Gamma(2)$-quasimodular in τ.

Nicest example: $F_{g=1} = -\log \eta(\tau)$

Question: Relate this to our rationality.

Gromov-Witten / Stable quotients

Work of many people. For $S = \mathbb{P}^1 \times \mathbb{P}^1$ follows H. Lho.

Moduli of SQ of K_S produces invariants related to GW by mirror map

$$F_g^{SQ}(z_1, z_2) = \sum_{i,j} N^{SQ}_{g,(i,j)} z_1^i z_2^j$$

Then

$$F^{SQ}(\quad) = F^{GW}(Q \quad Q)$$

Then

$$F_g^{SQ}(z_1, z_2) = F_g^{GW}(Q_1, Q_2)$$

mirror map

Theorem (Lho) ① $F_g^{SQ}(z, z) \in \mathbb{C}[L^{\pm}, A_2]$ for $g \geq 2$

② $F_g^{SQ}(z, z)$ satisfies a holomorphic anomaly equation.

Write $X = (1 - 8z_1)^{-1/2} = 1 + 4z_1 + 24z_1^2 + \ldots$

Theorem (Lho): $F_g^{SQ}(z_1, z_2) \in \mathbb{C}[X_1^2, (1+X^2)^{-1}][[z_2]]$

Question: How is this related to our rationality?