Counting curves in space
maps or equations?

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Enumerative geometry is an ancient and very interesting topic that has been pushing algebraic geometry since the late 1800s. The last ~ 30 years have seen great developments, partially thanks to input from theoretical physics.

1. Given 3 generic circles in the plane, how many circles are tangent to the 3 of them?
   Ans: 8 (Apollonius’ problem – Ancient Greece)

2. How many lines does a smooth cubic surface contain?
   Ans: 27 (A. Cayley, G. Salmon – 1849)

3. How many conics does a generic quintic 3-fold contain?
   Ans: 609250 (S. Katz – 1986)
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Ans: $N_1 = N_2 = 1$, $N_3 = 12$, $N_4 = 620$, ...

$$N_d = \sum_{d_1 + d_2 = d} N_{d_1} N_{d_2} \left( d_1^2 d_2^2 \left( \frac{3d - 4}{3d_1 - 2} \right) - d_1^3 d_2 \left( \frac{3d - 4}{3d_1 - 1} \right) \right)$$

(Kontsevich – 1994)
One problem, two solutions

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Two ways to think of curves:

\[ f : \mathbb{P}^1 \to \mathbb{P}^2 \]
\[ [x : y] \mapsto [x : y : 0] \]
\[ \mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^2} \]
\[ \mathcal{I} = (z = 0) \]
Stable maps and Gromov-Witten theory

The first approach (curves=maps) lead to the development of Gromov-Witten theory in the 90s. Gromov-Witten theory uses the space of stable maps
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\[ \overline{M}_{g,m}(X, \beta) = \{(C, p_1, \ldots, p_m, f)\} \]

parametrizing maps \( f : C \rightarrow X \) from a nodal curve of genus \( g \) to \( X \) such that \( f_*[C] = \beta \in H_2(X) \) and distinct marked points \( p_1, \ldots, p_m \in C \).
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But sometimes the spaces \( \overline{M}_{g,m}(X, \beta) \) are very singular, sometimes they have strata with higher dimension than expected, etc.
This problem was solved with the introduction of virtual fundamental classes by Behrend-Fantechi and Li-Tian (around 95).
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\[[\overline{M}_{g,m}(X, \beta)]^{\text{vir}} \in H_{2\text{virdim}}(\overline{M}_{g,m}(X, \beta)).\]
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$$[\overline{M}_{g,m}(X, \beta)]^{\text{vir}} \in H_{2\text{virdim}}(\overline{M}_{g,m}(X, \beta)).$$

This homology class lives in degree equal to the expected dimension

$$\text{virdim} = (\dim(X) - 3)(1 - g) + \int_{\beta} c_1(X) + m.$$
Gromov-Witten invariants

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This leads us to a special case: when $X$ is a Calabi-Yau 3-fold ($c_1(X) = 0$; e.g. quintic 3-fold) the expected dimension is always 0 (for $m = 0$).
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$$Z^X_{\text{GW}} = \exp \left( \sum_{g,\beta} GW^X_{g,\beta} u^{2g-2} z^\beta \right).$$
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When \( X \) is a 3-fold it admits a virtual fundamental class \([I_n(X, \beta)]^{\text{vir}}\). If moreover \( X \) is Calabi-Yau, the expected dimension is zero and we define \( DT \) invariants

\[ DT^X_{n, \beta} = \int_{[I_n(X, \beta)]^{\text{vir}}} 1 \in \mathbb{Z}. \]
A picture
A Miró picture
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$$Z^X_{DT} = \frac{\sum_{n,\beta} DT^X_{n,\beta} q^n z^\beta}{\sum_{n \geq 0} DT^X_{n,0} q^n}.$$
Normalized DT invariants

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$$Z_{DT}^X = \frac{\sum_{n, \beta} DT_{n, \beta}^X q^n z^\beta}{\sum_{n \geq 0} DT_{n, 0}^X q^n}.$$

**Theorem (Behrend-Fantechi, Li)**

For $\beta = 0$

$$\sum_{n \geq 0} DT_{n, 0}^X q^n = \prod_{k \geq 1} (1 - (-q)^k)^{-k \cdot e(X)}.$$
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**Definition**

A stable pair is a sheaf $F$ of pure dimension 1 together with a map $\phi : \mathcal{O}_X \to F$ such that $\text{coker } \phi$ has dimension 0. Let $P_n(X, \beta)$ be the moduli of stable pairs with $n = \chi(F)$, $\beta = [\text{supp}(F)]$. 

Think of stable pairs as a curve together with points on the curve. If $C \subseteq X$ is smooth then stable pairs supported on $C$ are $\mathcal{O}_X \to \mathcal{O}_C(D)$ with $D \subseteq C$ effective divisor.
Stable pairs and Pandharipande-Thomas invariants

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As before we define the PT invariants and the PT partition function:

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\text{PT}_n^X, \beta = \int [P_n(X, \beta)]_{\text{vir}} 1 \in \mathbb{Z}.
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Stable pairs have a striking rationality property:

**Theorem (Bridgeland 2016)**

*For every* \( \beta \in H_2(X) \) *the generating function*

\[ \PT_{\beta}^X = \sum_{n \in \mathbb{Z}} \PT_{n,\beta}^X q^n \]

*is the Laurent expansion of a rational function satisfying the symmetry*

\[ \PT_{\beta}^X(q) = \PT_{\beta}^X(q^{-1}). \]
Maps/equations correspondences

All the 3 enumerative theories discussed (GW, DT, PT) are expected to be equivalent.

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For a Calabi-Yau 3-fold $Z$

$$X_{DT} = X_{PT}.$$ 

The equivalence with Gromov-Witten is more complicated and still conjectural:

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande 2006)
After the change of variables $-q = e^{iu}$ we have

$$Z_{X_{GW}}(u, z) = Z_{X_{PT}}(-e^{iu}, z).$$ 

This opens a very interesting direction: we can use the equations side to study/compute the maps side!
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- Defined for every dimension (PT and DT only dimension 3).
- Are symplectic invariants.
- Defined over moduli of curves $M_{g,n}$.
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  - No multiple cover contributions.
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Let $X$ be a Calabi-Yau 3-fold containing a smooth divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $B \in H_2(X)$ be the curve class of $\mathbb{P}^1 \times \text{pt}$ (and assume the ray generated by $B$ is extremal in the curve cone of $X$).
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**Theorem (Buelles-M, 2021)**

Let $\beta \in H_2(X)$, $g \geq 0$. Assume GW/PT correspondence holds. Then

$$\sum_{j \in \mathbb{Z}} GW_{g,\beta + jB}^X Q^j$$

is the expansion of a rational function $f(Q)$ satisfying

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Suggested by physics as consequence of heterotic string+mirror symmetry.
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The symmetry is explained by a certain automorphism in the derived category

$$\rho = \text{ST}_{\mathcal{O}_E(-C)} \circ \text{ST}_{\mathcal{O}_E(-C+B)} \circ \mathbb{D} \in \text{Aut}(D^b(X)).$$
Thank you!