# Virtual Fundamental Classes after Behrend-Fantechi

Notes for talk at DT Aussois (10/10/22)

## §1. Motivation

In the 90s people were looking for a formal definition of Gromov-Witten invariants

- Should satisfy some basic axioms / properties
- Deformation invariant
- Agree w/ MS predictions

Manin-Kontsevich: Define GW as integrals over

$$\overline{M}_{g,n}(X,\beta) = \{ \text{stable maps } f: C \longrightarrow X \}$$

Example: $\# \{ \deg d \text{ conics} \subseteq \mathbb{P}^2 \text{ passing through } 3d-1 \text{ pts} \}$

$$\| \\ \int_{\overline{M}_{0,3d-1}(\mathbb{P}^2,d)} ev_1^*(pt) \cdots ev_{3d-1}^*(pt)$$

⚠ Problem: $\overline{M}_{g,n}(X,\beta)$ are not smooth and have the "wrong" dimension most of the time

2 solutions

Perturb the data to make $M$ smooth
(differential geometry)

Define a VFC $[M]^{vir} \in A_{vd}(M) \to H_{2vd}(M)$
(Li-Tian & Behrend-Fantechi)

Moduli space $M$ ⇝ Obstruction theory $\mathbb{L}_X \to E^\bullet$ ⇝ VFC $[M]^{vir}$ ⇝ Enumerative invariants $\int_{[M]^{vir}} \dots$

## §2. Local model

(From appendix of "Pandharipande-Thomas, 13/2 ways of counting curves")

$$\begin{tikzcd} & E \arrow[d] \\ M = Z(s) \arrow[r, hook] & A \arrow[u, bend right, "s"'] \end{tikzcd}$$

v.b. rank $r$

section

smooth dimension $n$

zero locus of $s$

If $s$ is ($C^\infty$) generic it is transverse to 0 section and $M = Z(s)$ is smooth of dim $n-r$.

We would like to define $[M]^{vir} \in A_{n-r}(M)$ for any $s$!

rank $r'$ v.b.

Simple case: $E$ splits as $\boxed{E'} \oplus E/E'$ and

$s = (s', 0)$ with $s'$ transverse to $0$-section of $E'$

$\rightsquigarrow$ $M$ is smooth of dim $n - r'$

$C^\infty$- generic

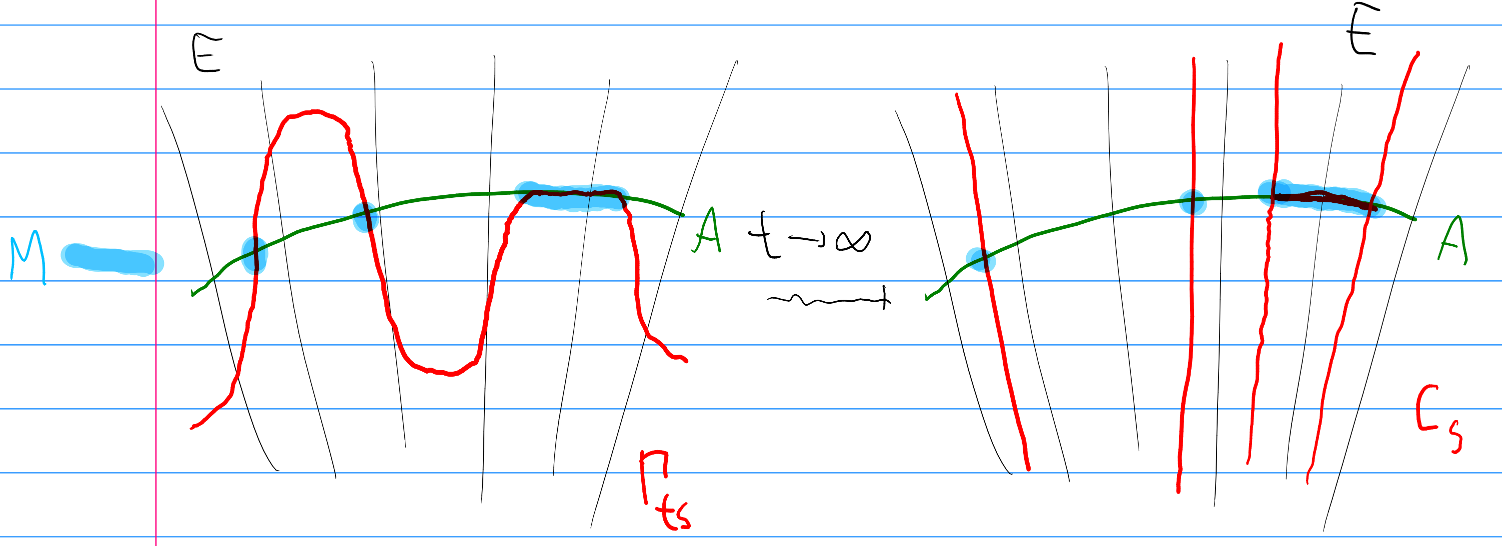
Perturb $s$ to $(s, \varepsilon)$ gives

$$Z(s,\varepsilon) \subseteq Z(s) = M$$

$$[M]^{vir} := [Z(s,\varepsilon)] = c_{r-r'}\underbrace{(E/E')}_{\text{"Obstruction bundle"}}$$

Without splitting: Use Fulton's refined intersection

$$[M]^{vir} = 0^!_E\,[C_s] \in A_{n-r}(M)$$



$C_s \subseteq E|_M$ is the limit of the graph $\Gamma_{ts}$ when $t \to \infty$

"obstruction cone"

**Fact:** $C_s$ is equidimensional of dim $n$

⚠ Such description almost never exists (unless we allow $A$ to be $\infty$-dimensional)

**Behrend - Fantechi:** Construct an intrinsic version of $C_s \subseteq E_{|M}$ using only "infinitesimal information"

$$0 \to T_pM \to \underbrace{T_pA \xrightarrow{ds} E_p}_{\text{Obstruction theory}} \to Ob_p \to 0$$

## §3. Obstruction theories

### • 3.1 Cotangent complex

$\Omega_X = T_X^\vee$ cotangent sheaf (= sheaf of Kähler differentials)

If $f: X \to Y$,

$$\underset{\text{would like to complete}}{?} \to f^*\Omega_Y \to \Omega_X \to \Omega_f \to 0$$

$$\underset{\overset{\cap}{D(X)}}{\mathbb{L}_X} = \text{"left-derived functor" of } \Omega_X$$

- $h^0(\mathbb{L}_X) = \Omega_X$
- $h^{>0}(\mathbb{L}_X) = 0$
- $f^*\mathbb{L}_X \to \mathbb{L}_Y \to \mathbb{L}_f$ is exact
- If $X \xhookrightarrow{\text{regular embedding}} \underset{\text{smooth}}{\textcircled{A}} \rightsquigarrow \mathbb{L}_X = [\overset{-1}{I/I^2} \to \overset{0}{\Omega_{A|X}}]$

  $I$ = ideal sheaf of $X \subseteq A$

## 3.2 Obstruction theory

**Def:** A obstruction theory on a scheme/DM stack $M$ is $E^\bullet \in D(M)$ together with a map $\phi : E^\bullet \to \mathbb{L}_M$ s.t.

- $h^0(E^\bullet) \xrightarrow{\sim} \underset{\substack{\wr \\ \Omega_M}}{h^0}(\mathbb{L}_M)$ iso
- $h^{-1}(E^\bullet) \twoheadrightarrow h^{-1}(\mathbb{L}_M)$ onto

**Rmk:** Moduli spaces often come with a natural obstruction theory

$TM = \mathrm{Mor}(\mathbb{C}[\varepsilon]/\varepsilon^2, M)$ = moduli description

**Ex:** $M = \mathrm{Maps}(C, X)$ where $C$ is fixed curve

$TM = H^0(C, f^*T_X) \rightsquigarrow E^\bullet = (R\pi_* f^*T_X)^\vee$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ M & \end{tikzcd}$$

**Def:** $E^\bullet$ is 2-term perfect if locally we can write

$$E^\bullet = [E^{-1} \to E^0]$$

for v.b. $E^{-1}, E^0$.

$E^\bullet$ perfect
$\Downarrow$
$rk(E^\bullet) = rk(E^0) - rk(E^{-1})$
is locally constant.

**Theorem** (Behrend-Fantechi, Li-Tian):

If $M$ has a 2-term pot there is a VFC

$$[M]^{vir} = [M, E^\bullet \xrightarrow{\phi} \mathbb{L}_M]^{vir} \in \begin{array}{c} A_{vd}(M) \\ \downarrow \\ H_{2vd}(M) \end{array}$$

$vd = rk(E^\bullet)$

**Rmk:** Some people call obstruction theory to the dual

$E_\bullet = (E^\bullet)^\vee$

$$h^0(E_\bullet) = TM$$

$$h^1(E_\bullet) =: Ob$$

"Obstruction bundle" or "Obstruction sheaf"

## §4. Construction of VFC

⚠ Very rough sketch!

**Stack:** "Space" that locally looks like quotient $[X/G]$.

Stacks keep information about $G$ even if it acts trivially ("record automorphisms")

**Ex:** $[pt/\mathbb{C}^\times]$ is a smooth stack of dimension $-1$.

**Def:** A v.b. stack over $M$ is a stack $\mathcal{V} \to M$ s.t. locally it looks like $[F/E]$ where $E \to F$ are vector bundle

cone stack $\qquad \mathcal{C} \to M$

$[C/E] \qquad E \to C \qquad E$ v.b.

$C$ cone

**Def/Prop:** Given a complex $L^{\bullet}$ s.t. $h^{>0}(L^{\bullet}) = 0$ there is a cone stack $c(L^{\bullet}) := h^1/h^0((L^{\bullet})^\vee)$ that locally looks as follows: If $L_{\bullet} = [L_0 \to L_1 \to \dots]$ is a free resolution on some (étale) open then

$$c(L^{\bullet}) = \left[\frac{\ker(L_1 \to L_2)}{L_0}\right]$$

If $L^\bullet$ is 2-term perfect then $c(L^\bullet)$ is a vb stack

The cone $c(L^\bullet)$ only depends on $L^\bullet$ up to quasi iso.

$$\begin{tikzcd} \text{Obs theory} \arrow[d, squiggly] & E^\bullet \arrow[r] \arrow[d, squiggly] & \mathbb{L}_M \\ \text{Morphism of cones} & c(\mathbb{L}^\bullet) \arrow[r] \arrow[d, equal] & c(E^\bullet) \arrow[d, equal] \\ & \mathfrak{N}_M & \mathcal{E} \end{tikzcd}$$

Intrinsic normal sheaf

**Proposition (BF):** $E^\bullet \to \mathbb{L}_M$ is obstruction theory iff $\mathfrak{N}_M \hookrightarrow \mathcal{E}$ is closed immersion

Tentative definition: $[M]^{vir} = 0^!_{\mathcal{E}}[\mathfrak{N}_M]$

⚠ Wrong! $\mathfrak{N}_M$ is not equidimensional.

How does $\mathfrak{N}_M$ look locally?

$$\begin{tikzcd} U \arrow[r, phantom, "\overset{\text{open}}{\subseteq}"] \arrow[d, hook, "\text{closed}"'] & M \\ A & \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{C}_{M|U} \arrow[r, hook] \arrow[d, equal] & \mathfrak{N}_{M|U} \arrow[d, equal] \\ {[C_{U/A}/T_{A|U}]} \arrow[r, hook] & {[N_{U/A}/T_{A|U}]} \end{tikzcd}$$

normal cone

Intrinsic normal cone

**Theorem (BF):** There is a subcone $\mathfrak{C}_M \subseteq \mathfrak{N}_M$ of pure dimension 0 that locally looks like above.

**Definition:** $$[M]^{vir} = 0^!_{\mathcal{E}}\,[\mathfrak{C}_M] \in A_{rk(E^\bullet)}(M)$$

$rk = -rk(E^\bullet)$

$\dim = 0$

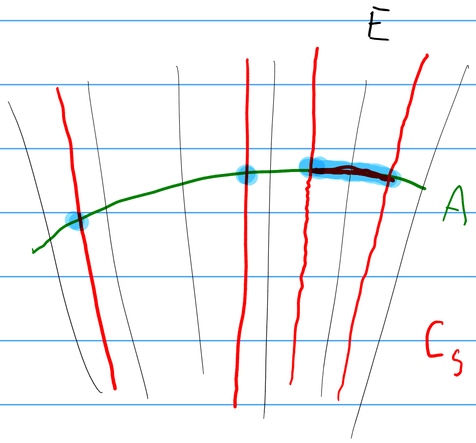
Uses Kresch's intersection thy for stacks

Back to local case:

$$\begin{tikzcd} & E \arrow[d] \\ Z(s) = M \arrow[r, hook] & A \arrow[u, bend right, "s"'] \end{tikzcd}$$

$$0 \to \underset{\overset{\|}{h^0(E_\bullet)}}{TM} \to \underbrace{TA|_M \xrightarrow{ds} E|_M}_{E_\bullet = (E^\bullet)^\vee} \to \underset{\overset{\|}{h^1(E^\bullet)}}{Ob_M} \to 0$$

$$\begin{tikzcd} C_s \arrow[r, hook] \arrow[d] & E|_M \arrow[d] \\ {[C_s / TA|_M]} \arrow[r, hook] \arrow[d, equal] & {[E|_M / TA|_M]} \arrow[d, equal] \\ \mathfrak{C}_M & \mathcal{E} \end{tikzcd}$$



**Rmk:** A 2-term pot on quasi-projective DM stack always has a global resolution $E_\bullet = [E_0 \to E_1]$

(Behrend)

The definition of $[M]^{vir}$ can be rewritten as

$$[M]^{vir} = 0^!_{E_0}[C]$$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_0 \arrow[d] \\ \mathcal{C}_X \arrow[r, hook] & \mathcal{E} \end{tikzcd}$$

Doesn't use intersection theory of stacks (was BF original definition)

"Obstruction cone"

A global resolution gives $E_1 \twoheadrightarrow Ob = E_1/E_0$

More generally, given $\Omega \twoheadrightarrow Ob$ we can construct an obstruction cone $C_\Omega \subseteq \Omega$

$$[M]^{vir} = 0^!_\Omega [C_\Omega]$$

pure dimension $\mathrm{rk}\,\Omega + \mathrm{rk}\,E^0$

## §5. Properties & tools

**Proposition:** Suppose $M$ is smooth and has pot $E^\bullet \to \mathbb{L}_X$

Then $Ob = h^1(E_\bullet)$ is a v.b. of rank $\dim M - vd = h^0(E_\bullet) - \mathrm{rk}(E_\bullet)$

$$[M]^{vir} = c_{top}(Ob) \cap [M]$$

**Theorem** (Manolache) If $f: M \to N$ has a relative 2-term pot $E_f^\bullet \to \mathbb{L}_f^\bullet$ then there is a virtual pullback

$$f^! : A_*(N) \to A_*(M)$$

If $E_M^\bullet$, $E_N^\bullet$ are compatible then

$$f^![N]^{vir} = [M]^{vir}$$

**Corollary**: "VFCs are deformation invariant"

Family of moduli spaces $\begin{matrix}\mathcal{M}\\ \downarrow \\ S\end{matrix}$ + family of pot give $[\mathcal{M}]^{vir} \in A_*(\mathcal{M})$ s.t. $[M_s]^{vir} = i_s^![\mathcal{M}]^{vir}$

$$\Rightarrow \int_{[M_s]^{vir}} i_s^* \alpha = \int_{[\mathcal{M}]^{vir}} \alpha \quad \text{is constant in } s.$$

**Theorem** (Graber – Pandharipande) Suppose a torus $T$ acts on $M$ and the obstruction theory is $T$-equivariant

$$[M]^{vir} = i_*\left(\frac{[M^T]^{vir}}{e(N^{vir})}\right) \qquad \in A_{loc}^T(X)$$

**Theorem** (Kiem-Li) Suppose there is a cosection $\mathcal{O}b \xrightarrow{\sigma} \mathcal{O}_M$. Let $Z \subseteq M$ be the locus where $\sigma$ is surjective $(Z = Z(\sigma^\vee))$. Then there is a localized VFC $[M]^{vir}_{loc} \in A_*(Z)$ s.t.

$$[M]^{vir} = i_*[M]^{vir}_{loc}$$

If $\sigma : \mathcal{O}b \twoheadrightarrow \mathcal{O}_M$ then $Z = \emptyset$ and $[M]^{vir} = 0$.

In this case there is a reduced VFC

$$[M]^{red} \in A_{vd+1}(M)$$