HIGHER RANGE ASYMPTOTIC FORMULAS FOR REFINED POINCARÉ POLYNOMIALS

MIGUEL MOREIRA

Throughout this appendix we denote by $M_{d,\chi}$ the moduli space of 1-dimensional stable sheaves with support of degree d and Euler characteristic χ, with the assumption that $\gcd(d,\chi)=1$. Our goals are:

- Explaining the relation between the asymptotic formulas for the Betti numbers found in the present paper and the approach to calculate the cohomology rings of moduli spaces $M_{d,\chi}$ in [PS23, KLMP24] (Corollary 0.3).
- Proposing a conjecture strengthening Conjecture 1.9 by allowing a larger range (Conjecture 0.4).
- Proposing a conjecture concerning the refinements coming from the perverse/Chern filtration (Conjecture 0.5).

0.1. **Generators and relations description of $H^*(M_{d,\chi})$.** The natural generators of the cohomology are the tautological classes, obtained by taking Kunneth components of Chern classes of the universal bundle. More precisely, we define

$$c_k(j) = p_*\big(\mathrm{ch}_{k+1}(\mathbb{F})q^*H^j\big) \in H^{2(k+j-1)}(M_{d,\chi})$$

for $k \geq 0$ and $j = 0,1,2$. Here, p, q are the projections of $M_{d,\chi} \times \mathbb{P}^2$ onto $M_{d,\chi}$ and $\mathbb{P}^2$, respectively, and $\mathbb{F}$ is a (rational, normalized) universal sheaf on the product $M_{d,\chi} \times \mathbb{P}^2$. The universal sheaf $\mathbb{F}$ is normalized in a way that $c_1(1) = 0$; we refer to [PS23, Section 1.1] for more details on the normalization. A classical argument by Beauville shows that tautological classes generate the cohomology:

Proposition 0.1 ([PS23], [Bea95])**.** *The classes $c_k(j)$ generate $H^*(M_{d,\chi})$ as a $\mathbb{Q}$-algebra.*

In other words, the natural algebra homomorphism from the free polynomial algebra

$$\mathbb{D} \coloneqq \mathbb{Q}[c_2(0), c_0(2), c_3(0), c_2(1), c_1(2), \ldots] \to H^*(M_{d,\chi}) \tag{1}$$

is surjective. The algebra $\mathbb{D}$ is sometimes referred to as the descendent algebra. The descendent algebra has a natural cohomological grading making the homomorphism above respect gradings. Note that $\mathbb{D}$ is infinite dimensional, but the subspaces $\mathbb{D}^{2j}$ of degree $2j$

Date: March 3, 2025.

elements are finite dimensional, and we have[1]

$$H(y) \coloneqq \sum_{j\geq 0} y^j \dim \mathbb{D}^{2j} = \prod_{k>0} \frac{1}{(1-y^k)^2(1-y^{k+1})}\,.$$

In [PS23, KLMP24] the authors construct families of geometric relations among tautological classes in $H^*(M_{d,\chi})$. The most important relations are the so called generalized Mumford relations. They come from the following simple observation [KLMP24, Proposition 2.6]: if F, F' are semistable sheaves with topological types (d,χ) and (d',χ') satisfying

$$\frac{\chi'}{d'} < \frac{\chi}{d} < \frac{\chi'}{d'} + 3 \tag{2}$$

then $\mathrm{Hom}(F,F') = \mathrm{Ext}^2(F,F') = 0$. Hence, $\mathrm{Ext}^1(F,F')$ has constant dimension dd', and therefore there is a vector bundle $\mathcal{V}$ of rank dd' on the product $M_{d,\chi} \times M_{d',\chi'}$ whose fiber over (F,F') is $\mathrm{Ext}^1(F,F')$. The generalized Mumford relations are obtained from the vanishing of Chern classes beyond the rank of $\mathcal{V}$:

$$c_k(\mathcal{V}) = 0 \text{ in } H^*(M_{d,\chi} \times M_{d',\chi'}) \text{ for } k > dd'\,.$$

We get relations

$$\mathsf{GMR}_{d,\chi}^{d',\chi',k,A} \in \ker\left(\mathbb{D} \to H^*(M_{d,\chi})\right)$$

for each d,χ,d',χ' satisfying (2), $k > dd'$ and $A \in H^*(M_{d',\chi'})$ by integrating $c_k(\mathcal{V})$ along the Poincaré dual of A. The generalized Mumford relation above has cohomological degree

$$2(k + \deg(A) - ((d')^2 + 1)) \geq 2d'(d - d')$$

with equality when A is the unit class and $j = dd' + 1$.

There are 2 other families of geometric relations introduced in [KLMP24], but they only appear in cohomological degree quadratic in d. These families of geometric relations do not necessarily generate the ideal of relations – for example, in $M_{5,1}$ there are three relations in degrees 36, 38 which are not in the ideal of geometric relations.

0.1.1. *The $H^{<2(d-1)}$ range.* The first generalized Mumford relations that appear are when $d' = 1$ or $d' = d-1$. The cohomological degree of GMR relations in either case starts at $2(d-1)$. Indeed, there are no relations among generators below such degree since, by the result of Yuan [Yua23],

$$\dim H^{2j}(M_{d,\chi}) = [y^j]H(y) = \dim \mathbb{D}^{2j}$$

for $j \leq d-2$.

[1]This is almost the same infinite product that appears in Theorem 1.6, except that the exponent in $1/(1-y)$ is 2 instead of 1. The difference is explained by the fact that we are not dividing by $\frac{1-y^{3d}}{1-y}$.

0.1.2. *The $H^{<2(2d-4)}$ range.* In the range between $2(d-1)$ and $2(2d-4)$ we have the effect of generalized Mumford relations, but only with[2] $d'=1$. When we take $\chi=1$ and $d'=1$, the inequality (2) is equivalent to $\chi' \in \{0,-1,-2\}$. Moreover, $M_{1,\chi'}$ is isomorphic to $\mathbb{P}^2$ (or, more canonically, it is isomorphic to the dual of the original $\mathbb{P}^2$), so A is taken from $\{h^i \colon i \in \{0,1,2\}\}$ where $h \in H^2(M_{1,\chi'})$ is the generator.

If we assemble into a generating series the degrees of all the possible generalized Mumford relations with fixed $d, \chi, d'=1$ and k, χ', A varying through the possibilities discussed, we get

$$\sum_{k>d} \sum_{\chi'=-2}^{0} \sum_{i=0}^{2} y^{k-2+i} = 3y^{d-1}\frac{(1+y+y^2)}{1-y}\,.$$

This is exactly the rational function appearing in the asymptotic formula of Theorem 1.6. The following proposition has been obtained in joint work of M.M. with Lim and Pi:

Proposition 0.2 (Lim–Pi–Moreira)**.** *The generalized Mumford relations with $d'=1$ are completely independent in $\mathbb{D}^{\leq 2(2d-4)}$. More precisely, if $j \leq 2d-4$ then the elements*

$$D \cdot \mathsf{GMR}_{d,\chi}^{d'=1,\chi',k,h^i} \in \mathbb{D}^{2j}$$

are linearly independent, where χ', k, i run through the possibilities discussed above and D runs through monomials in tautological classes with $\deg(D) + 2(k-2+i) = 2j$*.*

A consequence of the asymptotic formula of Theorem 1.6 is then the following:

Corollary 0.3. *For $j < 2d-10$ (or, assuming the better bound in Remark 1.8, $j < 2d-4$) the ideal of relations*

$$\ker\left(\mathbb{D}^{2j} \to H^{2j}(M_{d,\chi})\right)$$

is (freely) generated by the generalized Mumford relations with $d'=1$.

0.1.3. *Higher range.* When extending from $2(2d-4)$ to $2(3d-9)$, as in the proposed formula in Conjecture 1.9, two things happen. First, GMR relations with $d'=2$ play a role; secondly, there are redundancies in the relations. While this makes it hard to give such a simple explanation for the formula as in the $< 2(2d-4)$ range, let us rewrite Conjecture 1.9 in a form that is more natural.

This rewriting is inspired by the recursion for the Betti numbers of the moduli spaces of stable bundles on curves in [HN75, AB83], which can be explained via the Harder–Narasimhan stratification of the stack of all (not necessarily stable) vector bundles. Indeed, this stratification is closely related to generalized Mumford relations, and provides a way to study their redundancies, see for example [LMP24, Proposition 3.3].

[2] A priori, GMR with $d'=d-1$ could also play a role. However, it turns out that such relations are already contained in the ideal of GMR with $d'=1$ in the range considered.

Let $\widetilde{P}_{d,\chi}(y)$ be the Poincaré series of the stack $\mathfrak{M}_{d,\chi}$ of semistable sheaves, i.e.

$$\widetilde{P}_{d,\chi}(y) \coloneqq \sum_{j\geq 0} y^j \dim H^{2j}(\mathfrak{M}_{d,\chi}) \in \mathbb{Z}[\![y]\!] \,.$$

When $\gcd(d,\chi)=1$, $\mathfrak{M}_{d,\chi} \simeq M_{d,\chi} \times B\mathbb{C}^\times$ and

$$\widetilde{P}_{d,\chi}(y) = \frac{1}{1-y} P_d(y)\,.$$

In general, $\widetilde{P}_{d,\chi}$ depends only on d and $\gcd(d,\chi)$, and it can be recovered from knowing $P_{d/k}(y)$ for every k dividing $\gcd(d,\chi)$ by a plethystic exponential type formula, see [KLMP24, Corollary 4.9]. For example,

$$\begin{aligned}\widetilde{P}_{2,0}(y) &= \frac{P_2(y)}{1-y} + \frac{1}{2} y \left(\frac{P_1(y)}{1-y}\right)^2 - \frac{1}{2} y \frac{P_1(y^2)}{1-y^2} \\ &= \frac{(1+y+y^2)(1+y^2+y^3+y^4-y^5)}{(1-y)(1-y^2)}\,.\end{aligned}$$

The rewriting of Conjecture 1.9 depends on the following observation:

$$3f(y) = -3\widetilde{P}_{2,1} - 3\widetilde{P}_{2,0} + 6y\widetilde{P}_{1,0}^2\,.$$

In this formula, it is natural to interpret the term $3\widetilde{P}_{2,1}$ as coming from GMR with $d'=2$ and $\chi'=-1,-3,-5$ and $3\widetilde{P}_{2,0}$ coming from[3] GMR with $d'=2$ and $\chi'=0,-2,-4$. The term $6y\widetilde{P}_{1,0}^2$ might be interpreted as being related to redundancies among the relations. Then, Conjecture 1.9 is equivalent to the recursion

$$\begin{aligned} &P_d + 3y^{d-1}\widetilde{P}_{1,0}P_{d-1} + 3y^{2d-4}\widetilde{P}_{2,1}P_{d-2} \\ &\quad + 3y^{2d-4}\widetilde{P}_{2,0}P_{d-2} + 3y^{2d-3}\widetilde{P}_{1,0}^2P_{d-2} = H(y) \mod y^{3d-9}\,. \end{aligned} \tag{3}$$

This way of rewriting Conjecture 1.9 actually leads us to propose a more general conjecture. To state it, let HN_k be the set of "Harder–Narasimhan types", consisting of pairs $(\boldsymbol{d},\boldsymbol{\chi})$ where $\boldsymbol{d}=(d_1,\ldots,d_m)\in\mathbb{Z}_{>0}^m$ and $\boldsymbol{\chi}=(\chi_1,\ldots,\chi_m)\in\mathbb{Z}^m$ satisfy

$$d_1+\ldots+d_m \leq k \text{ and } 0 \leq \frac{\chi_1}{d_1} < \ldots < \frac{\chi_m}{d_m} < 3\,.$$

Set $d_0 = d - \sum_{i=1}^m d_i$ and let

$$s(\boldsymbol{d}) \coloneqq \sum_{0\leq i<j\leq m} d_i d_j\,.$$

Conjecture 0.4. *Let $k\geq 0$. For $d>k+1$ we have*

$$\sum_{(\mathbf{d},\boldsymbol{\chi})\in\mathsf{HN}_\mathbf{k}} y^{s(\boldsymbol{d})} P_{d_0}\widetilde{P}_{d_1,\chi_1}\ldots\widetilde{P}_{d_m,\chi_m} = H(y) \mod y^{(k+1)(d-k-1)}\,.$$

where the sum runs over the set of "Harder–Narasimhan types".

[3]Generalized Mumford relations can also be defined when $\gcd(d',\chi')\neq 1$, as long as one works with the stack of semistables.

The conjecture implies the existence of rational functions $f_0, f_1, \ldots$ with the property that

$$P_d(y) = H(y)\left(\sum_{j=0}^{k} y^{j(d-j)} f_j(y)\right) \mod y^{(k+1)(d-k-1)}.$$

for every $d > k+1$. The rational functions f_k only have poles at roots of unity. Indeed, it follows from [KLMP24, Theorem 4.11] that the denominator of f_k is $(1-y)\ldots(1-y^k)$. The rational function f_k can explicitly determined from $P_1, \ldots, P_k$. The first ones f_0, f_1, f_2 are already implicit in Conjecture 1.9. The next one is

$$f_3(y) = \frac{-3}{(1-y)(1-y^2)(1-y^3)}(9 + 18y - 44y^3 - 82y^4 - 37y^5 + 56y^6 + 143y^7 + 170y^8 + 164y^9 + 125y^{10} + 89y^{11} + 55y^{12} + 36y^{13} + 18y^{14} + 9y^{15}).$$

Conjecture 0.4 suggests that the generalized Mumford relations are complete at least up to cohomological degree $2\lfloor d^2/4 \rfloor - 1$.

0.2. **Chern/perverse refinement.** One of the reasons why the study of the cohomology of $M_{d,\chi}$ is interesting is that it gives direct access to Gopakumar–Vafa invariants of $K\mathbb{P}^2$, according to a proposal by Maulik–Toda [MT18]. They propose the definition of such invariants in terms of the perverse filtration

$$P_0 H^*(M_{d,\chi}) \subseteq P_1 H^*(M_{d,\chi}) \subseteq P_2 H^*(M_{d,\chi}) \subseteq \ldots$$

on $M_{d,\chi}$ associated to the Hilbert–Chow morphism

$$M_{d,\chi} \to |dH|$$

sending a 1-dimensional sheaf to its fitting support. The perverse filtration determines the refined Poincaré polynomial

$$P_d^{\mathsf{ref}}(q,t) \coloneqq \sum_{i,j\geq 0} q^i t^j \dim \mathrm{gr}_j^P H^{i+j}(M_{d,\chi}),$$

and Maulik–Toda propose the definition of $n_{g,d}^{\mathsf{MT}}$ in terms of the $t=1$ specialization:

$$P_d^{\mathsf{ref}}(q,1) = \sum_{g\geq 0} n_{g,d}^{\mathsf{MT}} (q^{1/2} - q^{-1/2})^{2g}.$$

The Gromov–Witten/Gopakumar–Vafa correspondence is the conjecture that these agree with $n_{g,d}$ defined via the Gromov-Witten theory of $K\mathbb{P}^2$, or equivalently $P_d^{\mathsf{ref}}(q,1) = \mathcal{F}_d(q)$. From this point of view, the log/local correspondence can be phrased as a surprising relation between the specializations $q = t = y^{1/2}$ (which recovers $P_d(y)$) and $q = y, t = 1$ of P_d^{ref}.

It has been conjectured in [KPS23, KLMP24] that the perverse filtration matches another natural filtration called the Chern filtration $C_\bullet H^*(M_{d,\chi})$, which is defined in terms of tautological classes:

$$C_k H^*(M_{d,\chi}) \coloneqq \mathrm{span}\{c_{k_1}(j_1)\ldots c_{k_m}(j_m) \colon k_1 + \ldots + k_m \leq k\}.$$

Note that the perverse filtration can be defined in $\mathbb{D}$ exactly in the same way, and we let

$$H^{\mathsf{ref}}(q,t) \coloneqq \sum_{i,j\geq 0} q^i t^j \dim \mathrm{gr}_j^P \mathbb{D}^{i+j} = \prod_{k>0} \frac{1}{(1-q^{k-1}t^{k+1})(1-q^{k+1}t^{k-1})(1-q^{k+1}t^{k+1})}\,.$$

The $P=C$ conjecture has been shown [MSY25, PSSZ24] to hold in the free range of the cohomology, i.e.

$$P_\bullet H^{\leq 2d-4}(M_{d,\chi}) = C_\bullet H^{\leq 2d-4}(M_{d,\chi})\,.$$

This equality implies the asymptotic formula

$$P_d^{\mathsf{ref}}(q,t) = H^{\mathsf{ref}}(q,t) \mod (q,t)^{2d-2}\,.$$

From the point of view of the Chern filtration and the asymptotic formulas in the present paper, it becomes very natural to ask if there are refinements of the higher asymptotic formulas discussed here for the $q=t=y^{1/2}$ specialization. Numerical evidence[4] suggests that this is indeed the case. To state our refined conjecture, one defines the refined Poincaré polynomials $\widetilde{P}_{d,\chi}^{\mathsf{ref}}$ of the stacks, which can still be calculated from P_d^{ref} with a plethystic exponential type formula [KLMP24, Remark 6.11]. For example,

$$\begin{aligned}\widetilde{P}_{2,0}^{\mathsf{ref}}(q,t) &= \frac{P_2^{\mathsf{ref}}(q,t)}{1-qt} + \frac{1}{2}qt\left(\frac{P_1^{\mathsf{ref}}(q,t)}{1-qt}\right)^2 - \frac{1}{2}qt\frac{P_1^{\mathsf{ref}}(q^2,t^2)}{1-q^2t^2}\\ &= \frac{(1+t^2+t^4)(1+qt^3+t^6+q^2t^6-q^2t^8)}{(1-qt)(1-q^2t^2)}\,.\end{aligned}$$

We refer the reader to [Dav25] for details on the definition of the perverse filtration on the stacks $\mathfrak{M}_{d,\chi}$; the plethystic formula is a consequence of [Dav25, (56)].

Given $(\boldsymbol{d},\boldsymbol{\chi}) \in \mathsf{HN}_k$, we let

$$s_\pm(\boldsymbol{d},\boldsymbol{\chi}) \coloneqq s(\boldsymbol{d}) \pm \sum_{i=1}^m \chi_i\,.$$

Conjecture 0.5. *Let $k\geq 0$. For $d>k+1$ we have*

$$\sum_{(\boldsymbol{d},\boldsymbol{\chi})\in\mathsf{HN}_k} q^{s_+(\boldsymbol{d},\boldsymbol{\chi})} t^{s_-(\boldsymbol{d},\boldsymbol{\chi})} P_{d_0}^{\mathsf{ref}} \widetilde{P}_{d_1,\chi_1}^{\mathsf{ref}} \dots \widetilde{P}_{d_m,\chi_m}^{\mathsf{ref}} = H^{\mathsf{ref}}(q,t) \mod (q,t)^{2(k+1)(d-k-1)}\,.$$

where the sum runs over the set of "Harder-Narasimhan types".

As in the unrefined case, the conjecture is equivalent to asymptotic formulas of the form

$$P_d^{\mathsf{ref}}(q,t) = H^{\mathsf{ref}}(q,t)\left(\sum_{j=0}^k q^{j(d-j)} t^{j(d-j-3)+1} f_j^{\mathsf{ref}}(q,t)\right) \mod (q,t)^{2(k+1)(d-k-1)}$$

[4]M.M. thanks Y. Kononov and W. Pi for sharing the (conjectural) data for the refined Poincaré polynomials, calculated via Nekrasov partition functions as in [KPS23, Section 3.4].

for every $d > k+1$, where $f_0^{\mathsf{ref}} = 1, f_1^{\mathsf{ref}}, f_2^{\mathsf{ref}}, \ldots$ are rational functions in q, t. The rational function f_k^{ref} can be explicitly calculated from $P_1^{\mathsf{ref}}, \ldots, P_k^{\mathsf{ref}}$. For example,

$$f_1^{\mathsf{ref}}(q,t) = -\frac{(q^2+qt+t^2)(1+t^2+t^4)}{1-qt}$$

$$f_2^{\mathsf{ref}}(q,t) = -\frac{(q^2+qt+t^2)(1+t^2+t^4)}{1-qt}\big(q^3+t^3-q^3t^2-qt^4-q^5t^2-q^3t^4-2q^2t^5 - q^4t^5-q^2t^7-qt^8-q^3t^8-q^5t^8+t^9-q^2t^{11}\big)\,.$$

Remark 0.6. Theorem 1.3 does not have a refined version. For example, the polynomials $P_d^{\mathsf{ref}}(q,t)$ are irreducible for $d = 3, 6, 9, 12$. On the other hand, they are divisible by t^4+t^2+1 for $d \leq 12$ not a multiple 3 (and the quotient is irreducible for $d \neq 1, 2$). It is natural to speculate that this divisibility holds in general.

References

[AB83] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[Bea95] A. Beauville. Sur la cohomologie de certains espaces de modules de fibrés vectoriels. In *Geometry and analysis (Bombay, 1992)*, pages 37–40. Tata Inst. Fund. Res., Bombay, 1995.

[Dav25] B. Davison. BPS Lie algebras and the less perverse filtration on the preprojective CoHA. *Adv. Math.*, 463: Paper No.110114, 2025.

[HN75] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.*, 212:215–248, 1974/75.

[KLMP24] Y. Kononov, W. Lim, M. Moreira, and W. Pi. Cohomology rings of the moduli of one-dimensional sheaves on the projective plane, preprint, 2024.

[KPS23] Y. Kononov, W. Pi, and J. Shen. Perverse filtrations, Chern filtrations, and refined BPS invariants for local $\mathbb{P}^2$. *Adv. Math.*, 433:Paper No. 109294, 29, 2023.

[LMP24] W. Lim, M. Moreira, and W. Pi. On the chern filtration for the moduli of bundles on curves, preprint, preprint, 2024.

[MSY25] D. Maulik, J. Shen, and Q. Yin. Perverse filtrations and fourier transforms, 2025, to appear in Acta Mathematica.

[MT18] D. Maulik, Y. Toda. Gopakumar-Vafa invariants via vanishing cycles. *Invent. Math.*, 213(3):1017–1097, 2018.

[PS23] W. Pi and J. Shen. Generators for the cohomology ring of the moduli of one-dimensional sheaves on $\mathbb{P}^2$. *Algebr. Geom.*, 10(4):504–520, 2023.

[PSSZ24] W. Pi, J. Shen, F. Si, and F. Zhang. Cohomological stabilization, perverse filtrations, and refined bps invariants for del Pezzo surfaces, preprint, 2024.

[Yua23] Y. Yuan. Sheaves on non-reduced curves in a projective surface. *Sci. China Math.*, 66(2):237–250, 2023.

Massachusetts Institute of Technology, Department of Mathematics
Email address: `miguel@mit.edu`