MODEL ANSWERS TO HWK #9

1. (iv) If we write $\vec{F}(x,y) = M\hat{\imath}+N\hat{\jmath}$, the curl of $\vec{F}$ is $\operatorname{curl}\vec{F} = N_x - M_y$. Since

$$N_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} \qquad \text{and} \qquad M_y = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2},$$

the curl of $\vec{F}$ vanishes everywhere $\vec{F}$ is defined.
(v) Let C_1 and C_2 be the curves in part (iii). Since

$$\int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} \neq \int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r},$$

the vector field $\vec{F}$ is not conservative over its entire domain. However, it is conservative over the right half plane $x > 0$ since $\theta_2 - \theta_1$ only depends on the endpoints and if we have a loop, we obviously get zero.
2. (i) Note that the gradient of $r = \sqrt{x^2+y^2}$ is

$$\nabla r = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\imath} = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath}.$$

The curl of $\vec{F}$ is then

$$\operatorname{curl}\vec{F} = \frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x) = nr^{n-1}\frac{x}{r}y - nr^{n-1}\frac{y}{r}x = 0.$$

(ii) Let $g(r) = \frac{1}{n+2}r^{n+2}$. Then

$$\nabla g(r) = r^n(x\hat{\imath} + y\hat{\jmath}),$$

so long as $n \neq -2$. If $n = -2$, consider instead $g(x,y) = \frac{1}{2}\ln(x^2+y^2)$; here

$$\nabla g(r) = \frac{1}{x^2+y^2}(x\hat{\imath}+y\hat{\jmath}) = \frac{1}{r}(x\hat{\imath}+y\hat{\jmath}).$$

3. (i) Consider the vector fields $\vec{F}_1 = \langle -y, 0\rangle$ and $\vec{F}_2 = \langle 0, x\rangle$. Both of these vector fields satisfy $\operatorname{curl}\vec{F} = 1$, and so applying Green's theorem to $\vec{F}_1$ gives

$$\operatorname{area}(R) = \int_R \mathrm{d}A = \int_R \operatorname{curl}\vec{F}_1\,\mathrm{d}A = -\oint_C y\,\mathrm{d}x,$$

and similarly to $\vec{F}_2$ gives

$$\operatorname{area}(R) = \int_R \mathrm{d}A = \int_R \operatorname{curl}\vec{F}_2\,\mathrm{d}A = \oint_C x\,\mathrm{d}y.$$

(ii) To obtain one arch we need the smallest positive t with $y(t) = 0$. This gives $t = 2\pi$. Let

$$\vec{r}_1(t) = \langle a(t - \sin t), a(1 - \cos t)\rangle \qquad \text{and} \qquad \vec{r}_2(t) = \langle 2\pi - t, 0\rangle.$$

Then the curve $C = C_1 \cup C_2$ encloses R with the opposite orientation, and so applying part (i) gives

$$\begin{aligned}
\operatorname{area}(R) &= -\int_C x\,\mathrm{d}y \\
&= -\int_{C_1} x\,\mathrm{d}y - \int_{C_2} x\,\mathrm{d}y \\
&= a^2 \int_0^{2\pi} \sin^2 t - t\sin t\,\mathrm{d}t \\
&= a^2 \left(\Big[t\cos t \Big]_0^{2\pi} - \int_0^{2\pi} \cos t\,\mathrm{d}t + \frac{1}{2}\int_0^{2\pi} (1 - \cos(2t))\,\mathrm{d}t \right) \\
&= 3\pi a^2.
\end{aligned}$$

4. (i) Observe that if

$$\vec{F} = \langle x^2y + y^3 - y, 3x + 2y^2x + e^y\rangle \qquad \text{then} \qquad \operatorname{curl}\vec{F} = 4 - (x^2 + y^2).$$

Therefore by Green's theorem we have that if C bounds the region R then

$$\oint_C \vec{F} \cdot \mathrm{d}r = \iint_R \operatorname{curl}\vec{F}\,\mathrm{d}A.$$

So we want R to be the region where the curl is at least zero, that is, we want $x^2 + y^2 \leq 4$. The boundary C of this region is the circle of radius 2, centred at the origin.

(ii) Again by applying Green's theorem we get that

$$\begin{aligned}
\int_C \vec{F} \cdot \mathrm{d}r &= \iint_R \operatorname{curl}\vec{F}\,\mathrm{d}A \\
&= \iint_{x^2+y^2\leq 4} (4 - x^2 - y^2)\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_0^2 (4 - r^2)r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi\left[2r^2 - \frac{1}{4}r^4\right]_0^2 \\
&= 8\pi.
\end{aligned}$$

5. (i) True. If $\vec{F} = \nabla f$ and $\vec{G} = \nabla g$ then $\vec{F} + \vec{G} = \nabla(f + g)$.

(ii) True. If $\vec{F}$ is a gradient vector field then $\operatorname{curl}\vec{F} = N_x - M_y = 0$. In particular $M_y(1,-1) = N_x(1,-1)$.

6. (i) Note that the normal vector to the unit circle is simply the radial vector $\langle x, y\rangle$. We compute the flux

$$\vec{F}\cdot\vec{n} = \langle xy, y^2\rangle \cdot \langle x, y\rangle = y(x^2+y^2).$$

We therefore see that $y \geq 0$, the upper half of the circle, contributes positively to the flux while $y \leq 0$, the lower half of the circle, contributes negatively to the flux.

(ii) Using the unit speed parametrization $\vec{r}(s) = \langle \cos s, \sin s\rangle$ with $s \in [0, 2\pi]$ we can use part (i) to compute

$$\begin{aligned}\int_0^{2\pi} \vec{F}\cdot\vec{n}\,\mathrm{d}s &= \int_0^{2\pi} \sin s(\cos^2 s + \sin^2 s)\,\mathrm{d}s \\ &= \int_0^{2\pi} \sin s\,\mathrm{d}s = 0.\end{aligned}$$

This gels with (i) because for each point (x, y) on the unit circle the flux at the corresponding point $(x, -y)$ has equal magnitude but opposite sign. Hence, we expect the total flux to be zero.

(iii) Using Green's theorem we get

$$\begin{aligned}\int_0^{2\pi} \vec{F}\cdot\vec{n}\,\mathrm{d}s &= \iint_{x^2+y^2\leq 1} \operatorname{div}\vec{F}\,\mathrm{d}A \\ &= \iint_{x^2+y^2\leq 1} 3y\,\mathrm{d}A = 0,\end{aligned}$$

since y is anti-symmetric about the x-axis and the unit circle is symmetric about the x-axis.