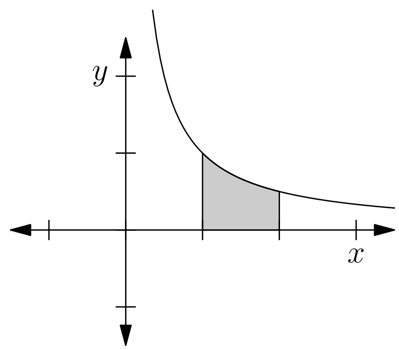
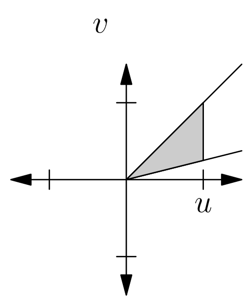
MODEL ANSWERS TO HWK #8

1. We will compute the polar moment of inertia of the region illustrated below, after changing coordinates to $u = xy$, $v = y/x$.



The tricky part is to set up the bounds for this region in the new coordinate system. Let's first describe the region in the uv-plane, and then choose the appropriate bounds of integration.
The condition $0 \leq xy \leq 1$ translates to $0 \leq u \leq 1$ in the new coordinates. On the other hand, $1 \leq x \leq 2$ translates to $1 \leq \sqrt{u/v} \leq 2$, so $v \leq u \leq 4v$. So the region is as illustrated:



It's clear that the bounds will be easier if the integral with respect to v is the inside one. We'll get

$$\int_{u=0}^{1} \int_{v=u/4}^{u} \text{something}(u, v)\, \mathrm{d}v\, \mathrm{d}u$$

To figure out exactly what something(u, v) is we need to compute the Jacobian and rewrite the function $x^2 + y^2$ in the coordinates u and v.

Since $x = \sqrt{u/v}$ and $y = \sqrt{uv}$, a bit of algebra shows that the Jacobian is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{\frac{u}{v}}v} & -\frac{u}{2\sqrt{\frac{u}{v}}v^2} \\ \frac{v}{2\sqrt{uv}} & \frac{u}{2\sqrt{uv}} \end{vmatrix} = \frac{1}{2v}.$$

Observe that the function $f(x,y) = x^2+y^2$ is given in these coordinates by $f(u,v) = uv + u/v$. Putting all of this together, the polar moment of inertia is given by

$$\int_{u=0}^{1}\int_{v=u/4}^{u} (uv + u/v)\,\delta(u,v)\,\frac{1}{2v}\,\mathrm{d}v\,\mathrm{d}u$$

One way to check this is to see that we get the same answer in both coordinate systems when we apply it to a uniform density function $\delta = 1$. It's easy to verify that

$$\int_{x=0}^{1}\int_{y=0}^{1/x} (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_{u=0}^{1}\int_{v=u/4}^{u} (uv + u/v)\left(\frac{1}{2v}\right)\mathrm{d}v\,\mathrm{d}u = \frac{13}{8},$$

so we've probably done everything right.

2. We're given an ellipse in the form $(2x+5y-3)^2+(3x-7y+8)^2 = 1$. Finding the area won't be so hard after we make the coordinate change $u = 2x+5y-3$ and $v = 3x-7y+8$.

The Jacobian is given by $\left|\det\left(\begin{smallmatrix} 2 & 5 \\ 3 & -7 \end{smallmatrix}\right)^{-1}\right| = 1/29$. So

$$\begin{aligned} A &= \iint_{(2x+5y-3)^2+(3x-7y+8)^2\le 1} 1\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{u^2+v^2\le 1} 1\left(\frac{1}{29}\right)\mathrm{d}u\,\mathrm{d}v = \frac{\pi}{29}, \end{aligned}$$

since $\iint_{u^2+v^2\le 1} 1\,\mathrm{d}u\,\mathrm{d}v = \pi$ is just the area of a circle of radius 1.

3. At each point (x,y), the slope of the vector $\vec{F}(x,y)$ is

$$\frac{4x}{1+x^2}.$$

A field line must be a curve whose tangent vector has this slope. So field lines are given by graphs of functions whose derivative is

$$\frac{4x}{1+x^2}.$$

Integrating shows that for each constant c, the curve

$$y = 2\ln(1+x^2) + c,$$

is a field line, and there are no other field lines.

4. We parameterise C as $\vec{r}(x) = x\hat{\imath} + f(x)\hat{\jmath}$, where $x_1 \leq x \leq x_2$. The line integral of $\vec{F}$ along C is then

$$\begin{aligned}
\int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_C \left(x^2 y + \frac{1}{3} y^3 \right) \hat{\imath} \cdot \left(\hat{\imath} + f'(x)\hat{\jmath} \right) \mathrm{d}x \\
&= \int_C \left(x^2 y + \frac{1}{3} y^3 \right) \mathrm{d}x \\
&= \int_{x_1}^{x_2} \left(x^2 f(x) + \frac{1}{3} f(x)^3 \right) \mathrm{d}x \\
&= \int_{x_1}^{x_2} \int_0^{f(x)} \left(x^2 + y^2 \right) \mathrm{d}y \, \mathrm{d}x,
\end{aligned}$$

this is precisely the polar moment of inertia $\iint_R r^2 \, \mathrm{d}A$.

5. (i) We compute:

$$\begin{aligned}
\nabla\theta(x, y) &= \nabla \left(\arctan \left(\frac{y}{x} \right) \right) \\
&= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \nabla \left(\frac{y}{x} \right) \\
&= \frac{x^2}{x^2 + y^2} \left(-\frac{y}{x^2} \hat{\imath} + \frac{1}{x} \hat{\jmath} \right) \\
&= \vec{F}(x, y).
\end{aligned}$$

(ii) Since $\vec{F}$ is a gradient vector field on the right half plane, and C is contained in the right half plane, the fundamental theorem of calculus for line integrals applies:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \theta(x_2, y_2) - \theta(x_1, y_1) = \theta_2 - \theta_1.$$

Alternatively, one may compute directly: if we write $\vec{r}(t) = x(t)\hat{\imath}+y(t)\hat{\jmath}$, then

$$\begin{aligned}
\int_C \vec{F}\cdot\vec{r} &= \int_C \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}\cdot(\hat{\imath}\,\mathrm{d}x+\hat{\jmath}\,\mathrm{d}y) \\
&= \int_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \\
&= \int_C \frac{-r\sin\theta}{r^2}\,\mathrm{d}(r\cos\theta) + \frac{r\cos\theta}{r^2}\,\mathrm{d}(r\sin\theta) \\
&= \int_C \frac{-r\sin\theta}{r^2}\,(\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta) + \frac{r\cos\theta}{r^2}\,(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta) \\
&= \int_C \mathrm{d}\theta,
\end{aligned}$$

and the final integral evaluates to $\theta_2 - \theta_1$.

(iii) Parameterise C_1 by $r(\theta) = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$, where $0 \leq \theta \leq \pi$. Then

$$\begin{aligned}
\int_{C_1} \vec{F}\cdot\mathrm{d}\vec{r} &= \int_{C_1} \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}\cdot(-\sin\theta\hat{\imath}+\cos\theta\hat{\jmath})\,\mathrm{d}t \\
&= \int_{C_1} \frac{y\sin\theta + x\cos\theta}{x^2+y^2}\,\mathrm{d}t \\
&= \int_0^\pi (\sin^2\theta + \cos^2\theta)\,\mathrm{d}t
\end{aligned}$$

and so $\int_{C_1} \vec{F}\cdot\mathrm{d}\vec{r} = \pi$.

If we parameterise C_2 by $r(\theta) = \cos\theta\hat{\imath} - \sin\theta\hat{\jmath}$, $0 \leq \theta \leq \pi$; we calculate

$$\begin{aligned}
\int_{C_2} \vec{F}\cdot\mathrm{d}\vec{r} &= \int_{C_1} \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}\cdot(-\sin\theta\hat{\imath}-\cos\theta\hat{\jmath})\,\mathrm{d}t \\
&= \int_{C_1} \frac{y\sin\theta - x\cos\theta}{x^2+y^2}\,\mathrm{d}t \\
&= \int_0^\pi (-\sin^2\theta - \cos^2\theta)\,\mathrm{d}t,
\end{aligned}$$

thus $\int_{C_2} \vec{F}\cdot\mathrm{d}\vec{r} = -\pi$.