MODEL ANSWERS TO HWK #7

1. We're doing an integral over the disk, and so it seems likely that working in polar coordinates is going to be the way to go. You could of course work in rectangular coordinates as well, but the integrals involved would be trickier. The function whose average we want to compute is the distance from the point to the centre of the circle, which is just $f(r, \theta) = r$. Using the fact that the area element in polar is $r\,\mathrm{d}r\,\mathrm{d}\theta$, and dividing through by the area of the circle πa^2, we compute

$$\begin{aligned} A &= \frac{1}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} f(r,\theta)\, r\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \frac{1}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \frac{1}{\pi a^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned}$$

The inner integral is

$$\int_{r=0}^{a} r^2\,dr = \left[\frac{r^3}{3}\right]_0^a = \frac{a^3}{3}.$$

and so the average distance is

$$\frac{1}{\pi a^2}\int_{\theta=0}^{2\pi}\frac{a^3}{3}\,\mathrm{d}\theta = \frac{2a}{3}.$$

2. The distance d from (x_0, y_0) to the line $x = \bar{x}$ is $|\bar{x} - x_0|$, and so $d(x, y)^2 = (x - \bar{x})^2$. We now use this fact to prove the identity:

$$\begin{aligned} \bar{I} &= \iint_R d^2\delta\,\mathrm{d}A \\ &= \iint_R (x-\bar{x})^2\delta\,\mathrm{d}A \\ &= \iint_R x^2\delta\,\mathrm{d}A - 2\bar{x}\iint_R x\delta\,\mathrm{d}A + \iint_R \bar{x}^2\delta\,\mathrm{d}A \\ &= I - 2\bar{x}(M\bar{x}) + \bar{x}^2(M) \\ &= I - M\bar{x}^2, \end{aligned}$$

whence $I = \bar{I} + M\bar{x}^2$, as claimed. You might have encountered this fact before in 8.01 or AP Physics C, where it's called the "parallel axis theorem".

3. The triangle has vertices at $(1,0)$, $(\cos\theta_1, \sin\theta_1)$, and $(\cos\theta_2, \sin\theta_2)$. Both angles vary from 0 to 2π.

We can compute the area as half of the absolute value of the determinant of a matrix whose rows are the edges of the triangle from $(0,0)$ to $(\cos\theta_1, \sin\theta_1)$ and $(0,0)$ to $(\cos\theta_2, \sin\theta_2)$ (this is the formula for the area of a parallelogram from earlier in the course). Those vectors are

$$\vec{v}_1 = \langle \cos\theta_1 - 1, \sin\theta_1 \rangle \qquad \text{and} \qquad \vec{v}_2 = \langle \cos\theta_2 - 1, \sin\theta_2 \rangle\,,$$

The area is given by

$$A = \frac{1}{2}\left|\det\begin{pmatrix} \cos\theta_1 - 1 & \sin\theta_1 \\ \cos\theta_2 - 1 & \sin\theta_2 \end{pmatrix}\right| = \frac{1}{2}\left|(\cos\theta_1 - 1)\sin\theta_2 - (\cos\theta_2 - 1)\sin\theta_1\right|.$$

Doing an integral with an absolute value in it can be difficult, and usually the right approach is to figure out on what region the quantity inside the absolute value is positive, on what region it is negative, and deal with these pieces separately. For us, if $\theta_1 \leq \theta_2$, then v_1 and v_2 are positively oriented, and the determinant is automatically positive. So we can break our region $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_2 \leq 2\pi$ into two regions, depending on whether $\theta_1 \leq \theta_2$. On $\theta_1 \leq \theta_2$, the area is equal to the above determinant

$$A(\theta_1, \theta_2) = \frac{1}{2}\left((\cos\theta_1 - 1)\sin\theta_2 - (\cos\theta_2 - 1)\sin\theta_1\right),$$

while if $\theta_1 \geq \theta_2$, the area is equal to the opposite of the determinant.

However, we can make a further simplifying observation, which is that the average is clearly equal on these two regions, so there's no point in working out the integral on both of them: it's enough to find the average area when $\theta_1 \leq \theta_2$. This integral takes place over the region $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$, which is a triangle whose area is $2\pi^2$.

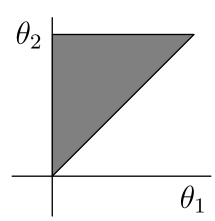

FIGURE 1. Region of integration

Now we just plug in our formula for the area and integrate:

$$\begin{aligned} A &= \frac{1}{2\pi^2}\int_{\theta_2=0}^{2\pi}\int_{\theta_1=0}^{\theta_2}\frac{1}{2}\left((\cos\theta_1-1)\sin\theta_2-(\cos\theta_2-1)\sin\theta_1\right)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\ &= \frac{1}{4\pi^2}\int_{\theta_2=0}^{2\pi}\int_{\theta_1=0}^{\theta_2}(\cos\theta_1-1)\sin\theta_2-(\cos\theta_2-1)\sin\theta_1\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \end{aligned}$$

So the inner integral is

$$\begin{aligned} &\int_{\theta_1=0}^{\theta_2}(\cos\theta_1-1)\sin\theta_2-(\cos\theta_2-1)\sin\theta_1\,\mathrm{d}\theta_1 \\ &= \Big[(\sin\theta_1-\theta_1)\sin\theta_2+(\cos\theta_2-1)\cos\theta_1\Big]_{\theta_1=0}^{\theta_2} \\ &= \sin^2\theta_2-\theta_2\sin\theta_2+(\cos\theta_2-1)\cos\theta_2-(\cos\theta_2-1) \\ &= 2-2\cos\theta_2-\theta_2\sin\theta_2. \end{aligned}$$

Therefore the outer integral is

$$\begin{aligned} \int_{\theta_2=0}^{2\pi}2-2\cos\theta_2-\theta_2\sin\theta_2\,\mathrm{d}\theta_2 &= \Big[2\theta_2-2\sin\theta_2+\theta_2\cos\theta_2-\sin\theta_2\Big]_{\theta_2=0}^{2\pi} \\ &= 6\pi. \end{aligned}$$

So the average is

$$\frac{3}{2\pi}\approx 0.477.$$

Since the area ranges between 0 and $3\sqrt{3}/4\approx 1.299$ (obtained for an equilateral triangle), this answer seems reasonable.

There are other ways to compute the average. The important thing is to divide up the region of integration for θ_1 and θ_2 into pieces for which you have a simple formula for the area in terms of the two angles.