MODEL ANSWERS TO HWK #6

1. Let $f(x,y) = x^2y$ and $g(x,y) = xy^2$. We are supposed to find the maximum and minimum of $f(x,y)$ subject to $g(x,y) = 5$, $x > 0$ and $y > 0$.

There are two ways to proceed. Here is the first. From the constraint $g(x,y) = 5$, we have $x = \frac{5}{y^2}$. Thus,

$$f(x,y) = f(\frac{5}{y^2}, y) = \frac{25}{y^3}.$$

The value of $f(x,y)$ approaches 0 as $y \to \infty$ along the constraint surface $xy^2 = 5$ and $f(x,y)$ grows to infinity as $y \to 0$ along the same surface. Thus there is no maximum and minimum.

Alternatively if we use the method of Lagrange multipliers, upon setting $\nabla f = \lambda \nabla g$, we would have to solve the following system:

$$\begin{aligned} 2xy &= \lambda y^2 \\ x^2 &= 2\lambda xy. \end{aligned}$$

Multiplying the first equation by $2x$, the second by y and subtracting we are led to

$$3x^2y = 0.$$

But then there is no solution to the above system of equations with both x and y non-zero.

2. (i) The expression

$$\left(\frac{\partial w}{\partial x}\right)_x$$

doesn't make sense, since we cannot both fix x and vary x.

(ii) Similarly, the expression

$$\left(\frac{\partial w}{\partial x}\right)_y$$

doesn't make sense, since if we fix y then x is fixed by the equation $g(x,y) = c$. Put differently, if we fix y and vary x, we are supposed to write z as a function of x; but z does not appear in the equation $g(x,y) = c$, so this does not make sense.

(iii) So the only expression in the list which makes sense is

$$\left(\frac{\partial w}{\partial x}\right)_z$$

which we can evaluate using the chain rule as follows:

$$\left(\frac{\partial w}{\partial x}\right)_z = f_x + f_y\left(\frac{\partial y}{\partial x}\right)_z + f_z\left(\frac{\partial z}{\partial x}\right)_z.$$

The last term is zero, and to evaluate the second, note that

$$0 = \left(\frac{\partial g}{\partial x}\right)_z = g_x + g_y\left(\frac{\partial y}{\partial x}\right)_z.$$

So

$$\left(\frac{\partial w}{\partial x}\right)_z = f_x - f_y\frac{g_x}{g_y}.$$

3. (i) Differentiating $t = \sin(x+y)$ with respect to t, fixing x, we get

$$1 = \left(\frac{\partial t}{\partial t}\right)_x = \cos(x+y)\left(\frac{\partial x}{\partial t}\right)_x + \cos(x+y)\left(\frac{\partial y}{\partial t}\right)_x = 0 + \cos(x+y)\left(\frac{\partial y}{\partial t}\right)_x.$$

That is,

$$\left(\frac{\partial y}{\partial t}\right)_x = \sec(x+y).$$

So

$$\left(\frac{\partial w}{\partial t}\right)_x = x^3y + x^3t\left(\frac{\partial y}{\partial t}\right)_x = x^3y + x^3t\sec(x+y).$$

(ii) We have

$$5x^4\,\mathrm{d}x + z\,\mathrm{d}y + y\,\mathrm{d}z = 0$$

and

$$(y^2 + 2zx)\,\mathrm{d}x + (2xy + z^2)\,\mathrm{d}y + (x^2 + 2yz)\,\mathrm{d}z = 0.$$

We may evaluate the coefficients of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ at $(x, y, z) = (1, 1, 2)$ to get the system

$$\begin{aligned} 5\,\mathrm{d}x + 2\,\mathrm{d}y + \mathrm{d}z &= 0 \\ 5\,\mathrm{d}x + 6\,\mathrm{d}y + 5\,\mathrm{d}z &= 0 \end{aligned}$$

Five times the first equation minus the second gives

$$20\,\mathrm{d}x + 4\,\mathrm{d}y = 0,$$

and so

$$\frac{dy}{dx} = -\frac{1}{5}$$

at the point $(1, 1, 2)$.

4. (i) We have

$$\int_1^a e^{-xy}\,\mathrm{d}y = \left[\frac{e^{-xy}}{-x}\right]_1^a = \frac{e^{-x} - e^{-ax}}{x}.$$

(ii) So our integral is

$$I = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\,\mathrm{d}x = \int_0^\infty \int_1^a e^{-xy}\,\mathrm{d}y\,\mathrm{d}x$$

Let's pretend that we're allowed to switch the order of integration of this improper integral (we are). Then

$$I = \int_1^a \int_0^\infty e^{-xy}\,\mathrm{d}x\,\mathrm{d}y.$$

The inner integral is

$$\int_0^\infty e^{-xy}\,\mathrm{d}x = \left[\frac{e^{-xy}}{-y}\right]_0^\infty = \frac{1}{y}.$$

So the outer integral is

$$\int_1^a \frac{1}{y}\,\mathrm{d}y = \Big[\ln y\Big]_1^a = \ln a.$$