MODEL ANSWERS TO HWK #5

1 (i) By the chain rule,

$$\begin{aligned} w_r &= w_x x_r + w_y y_r = w_x \cos\theta + w_y \sin\theta \\ w_\theta &= w_x x_\theta + w_y y_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta). \end{aligned}$$

Rewriting these equations in terms of matrices,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix} = \begin{pmatrix} w_r \\ w_\theta \end{pmatrix}.$$

(ii) By the chain rule,

$$\begin{aligned} w_x &= w_r r_x + w_\theta \theta_x = w_r\left(\frac{x}{\sqrt{x^2+y^2}}\right) + w_\theta\left(-\frac{y}{x^2+y^2}\right) \\ w_y &= w_r r_y + w_\theta \theta_y = w_r\left(\frac{y}{\sqrt{x^2+y^2}}\right) + w_\theta\left(\frac{x}{x^2+y^2}\right). \end{aligned}$$

Rewriting these equations in terms of matrices,

$$\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} \begin{pmatrix} w_r \\ w_\theta \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}.$$

(iii) First we change variables in B from x, y to θ, r. So,

$$B = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \frac{r\cos\theta}{r} & -\frac{r\sin\theta}{r^2} \\ \frac{r\sin\theta}{r} & \frac{r\cos\theta}{r^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}.$$

Then we compute,

$$AB = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $AB = I$ we know that $B = A^{-1}$.

(iv) We know from the above calculations that

$$B \begin{pmatrix} w_r \\ w_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} w_r \\ w_\theta \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}.$$

Plugging the values $w_r = 2$, $w_\theta = 10$, $r = 5$, and $\theta = \pi/2$ into the above equation yields

$$\begin{pmatrix} w_x \\ w_y \end{pmatrix} = \begin{pmatrix} 0 & -1/5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 10 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

2. Let $\hat{u} = \langle a, b\rangle$ and $\vec{r}(s) = \langle 1/2, 1\rangle + s\langle a, b\rangle$. Then

$$\left.\frac{df}{ds}\right|_{\hat{u}} = \frac{d}{ds}\big(f(\vec{r}(s))\big) = \frac{d}{ds}\big(f(\frac{1}{2} + sa, 1 + sb)\big) = \nabla f(\frac{1}{2}, 1) \cdot \langle a, b\rangle.$$

Since $\nabla f(x, y) = \langle 3x^2 - y^2 - 8x + 3 + 2xy, -2xy + x^2\rangle$, we have

$$\nabla f(\frac{1}{2}, 1) = \langle -\frac{1}{4}, -\frac{3}{4}\rangle.$$

Thus,

$$\left.\frac{df}{ds}\right|_{\hat{u}} = \langle \frac{-1}{4}, \frac{-3}{4}\rangle \cdot \langle a, b\rangle = |\langle \frac{-1}{4}, \frac{-3}{4}\rangle| \cos\theta = \frac{\sqrt{5}}{2\sqrt{2}} \cos\theta,$$

where θ is the angle between $\hat{u}$ and $\langle 1/2, 1\rangle$, and so

$$-\frac{\sqrt{5}}{2\sqrt{2}} \leq \left.\frac{df}{ds}\right|_{\hat{u}} \leq \frac{\sqrt{5}}{2\sqrt{2}}.$$

(i) and (ii) The right inequality is attained if the direction of $\hat{u}$ is the same as $\langle \frac{-1}{4}, \frac{-3}{4}\rangle$, that is, the maximum is achieved in the direction

$$\hat{u}_{\max} = \frac{\sqrt{2}}{\sqrt{5}}\langle \frac{-1}{2}, \frac{-3}{2}\rangle,$$

in which case the maximum value of the directional derivative $\left.\frac{df}{ds}\right|_{\hat{u}}$ is

$$\left.\frac{df}{ds}\right|_{\hat{u}_{\max}} = \frac{\sqrt{5}}{2\sqrt{2}}.$$

Similarly the minimum value is achieved in the direction

$$\hat{u}_{\min} = -\hat{u}_{\max} = \frac{\sqrt{2}}{\sqrt{5}}\langle \frac{1}{2}, \frac{3}{2}\rangle,$$

and the minimum value of the directional derivative $\left.\frac{df}{ds}\right|_{\hat{u}}$ is

$$\left.\frac{df}{ds}\right|_{\hat{u}_{\min}} = -\frac{\sqrt{5}}{2\sqrt{2}}.$$

(iii) The directional derivative is 0 in the two directions orthogonal to $\hat{u}_{\max}$,

$$\pm\frac{\sqrt{2}}{\sqrt{5}}\langle -\frac{3}{2}, \frac{1}{2}\rangle$$

3. (i) $\nabla g = \langle 2x, z, y\rangle$ so that at $(2 - 1, 1)$ we have $\langle 4, 1, -1\rangle$ and so we want the direction $\hat{u} = -\frac{1}{3\sqrt{2}}\langle 4, 1, -1\rangle$.

(ii) Let

$$\vec{r}(s) = \langle 2, -1, 1\rangle + \frac{s}{3\sqrt{2}}\langle -4, -1, 1\rangle.$$

The magnitude of $\langle 4, 1, -1\rangle$ is $3\sqrt{2}$. The value of g at $(2, -1, 1)$ is $g(2, -1, 1) = 3$. We want to decrease g by 1, so we want to move a distance $s = \frac{1}{3\sqrt{2}}$, that is, we want the point

$$\vec{r}(\frac{1}{3\sqrt{2}}) = \langle 2, -1, 1\rangle + \frac{1}{18}\langle -4, -1, 1\rangle = \langle 16/9, -19/18, 19/18\rangle.$$

Note that

$$g(\frac{16}{9}, -\frac{19}{18}, \frac{19}{18}) = \frac{221}{108} \approx 2.046.$$