# MODEL ANSWERS TO HWK #4

1. (i) From note LS we have

$$A = \begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{pmatrix} \qquad \text{and} \qquad \vec{r} = \begin{pmatrix} \sum x_i y_i \\ \sum y_i \end{pmatrix}.$$

Now

$$\begin{aligned} \sum x_i y_i &= \vec{x} \cdot \vec{y} = \vec{x} * \vec{y}' \\ \sum x_i^2 &= \vec{x} \cdot \vec{x} = \vec{x} * \vec{x}' \\ \sum x_i &= \vec{x} \cdot \vec{u} = \vec{x} * \vec{u}' \\ \sum y_i &= \vec{y} \cdot \vec{u} = \vec{y} * \vec{u}' \\ n &= \vec{u} \cdot \vec{u} = \vec{u} * \vec{u}', \end{aligned}$$

and so in Matlab notation we want

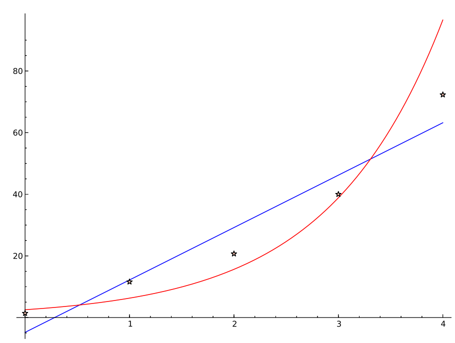
$$A = \begin{pmatrix} \vec{x} * \vec{x}' & \vec{x} * \vec{u}' \\ \vec{x} * \vec{u}' & \vec{u} * \vec{u}' \end{pmatrix} \qquad \text{and} \qquad \vec{r} = \begin{pmatrix} \vec{x} * \vec{y}' \\ \vec{y} * \vec{u}' \end{pmatrix}.$$

(ii)

(iii) $a = 17$, $b = -4.8$; the worse case is when $x = 4$ (so 2011). The actual value of $y$ is 72.3 whilst the best straight line predicts $y = 63.2$ an error of 9.1.

(iv) $a_1 = 0.91$, $b_1 = 0.93$; the worse case is when $x = 4$ (so 2011). The actual value of $y$ is 72.3 whilst the best exponential fit predicts $y = 96.5$ an error of 24.2.

(v) Straight line is blue, exponential is red.



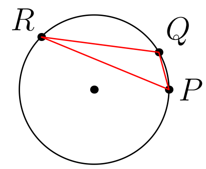



FIGURE 1. Triangle $PQR$

2. (i) Let $P = (1, 0)$, $Q = (\cos\theta_1, \sin\theta_1)$ and $R = (\cos\theta_2, \sin\theta_2)$, so $P$, $Q$ and $R$ are the vertices of the triangle in the $xy$-plane:

Since $\overrightarrow{PQ} = (\cos\theta_1 - 1)\hat{\imath} + \sin\theta_1\hat{\jmath}$ and $\overrightarrow{PR} = (\cos\theta_2 - 1)\hat{\imath} + \sin\theta_2\hat{\jmath}$ are two sides of the triangle, we know that the area $A$ of the triangle is given by

$$\begin{aligned} A(\theta_1, \theta_2) &= \frac{1}{2}\begin{vmatrix} \cos\theta_1 - 1 & \sin\theta_1 \\ \cos\theta_2 - 1 & \sin\theta_2 \end{vmatrix} \\ &= \frac{1}{2}(\sin\theta_2(\cos\theta_1 - 1) - \sin\theta_1(\cos\theta_2 - 1)). \end{aligned}$$

By assumption

$$0 \leq \theta_1 \leq \theta_2 \leq 2\pi,$$

and we may assume that $(\theta_1, \theta_2) \neq (0, 2\pi)$, $(2\pi, 2\pi)$ (since these are the same solutions as $(\theta_1, \theta_2) = (0, 0)$).

(ii) We compute,

$$\begin{aligned} A_{\theta_1} &= \frac{1}{2}(-\sin\theta_2\sin\theta_1 - \cos\theta_1(\cos\theta_2 - 1)), \\ A_{\theta_2} &= \frac{1}{2}(\cos\theta_2(\cos\theta_1 - 1) + \sin\theta_1\sin\theta_2). \end{aligned}$$

If we add these equations together and set both $A_{\theta_1}$ and $A_{\theta_2}$ equal to zero to find the critical points we get

$$0 = \cos\theta_2(\cos\theta_1 - 1) - \cos\theta_1(\cos\theta_2 - 1)$$

so that

$$0 = \cos\theta_1 - \cos\theta_2.$$

But then either $\theta_1 = \theta_2$ or $\theta_1 = 2\pi - \theta_2$, so that $\sin\theta_1 = \pm\sin\theta_2$. Setting $A_{\theta_1} = 0$ the top equation reduces to

$$\begin{aligned} \sin^2\theta_1 + \cos^2\theta_1 - \cos\theta_1 &= 0 && \text{if } \theta_1 = \theta_2 \\ -\sin^2\theta_1 + \cos^2\theta_1 - \cos\theta_1 &= 0 && \text{if } \theta_1 = 2\pi - \theta_2. \end{aligned}$$

The first equation reduces to $\cos\theta_1 = 1$ which has solution $(\theta_1, \theta_2) = (0, 0)$. The second equation reduces to

$$0 = 2\cos^2\theta_1 - \cos\theta_1 - 1 = (2\cos\theta_1 + 1)(\cos\theta_1 - 1),$$

which yields the additional solutions $(\theta_1, \theta_2) = (2\pi/3, 4\pi/3)$.

Therefore the critical points are $(0, 0)$ and $(2\pi/3, 4\pi/3)$.

(iii) The boundary of the region is $\theta_1 = 0$, $\theta_2 = 2\pi$ and $\theta_1 = \theta_2$.

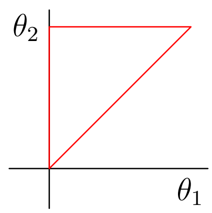



FIGURE 2. The $\theta_1\theta_2$-plane

The only critical point in the interior is $(2\pi/3, 4\pi/3)$.

$$A(2\pi/3, 4\pi/3) = \frac{3\sqrt{3}}{4},$$

and

$$0 = A(0, \theta_2) = A(\theta_1, 0) = A(\theta_1, \theta_1).$$

In fact the boundary cases all correspond to degenerate triangles whose area is zero. Therefore, the minimum of $A$ is 0 and the maximum of $A$ is $\frac{3\sqrt{3}}{4}$.

The minimum is achieved when two vertices coincide. Either $P = Q$ when $\theta_1 = 0$, $Q = R$, when $\theta_1 = \theta_2$ or $P = R$, $\theta_2 = 2\pi$. In all cases the area is zero. Note that $P = Q = R$, when $\theta_1 = \theta_2 = 0$; this is also a critical point of $A(\theta_1, \theta_2)$.

The maximum is achieved when the three vertices are $(0, 1)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. In this case we have an equilateral triangle.

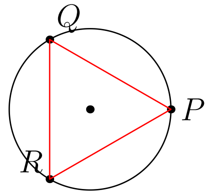



FIGURE 3. Equilateral triangle

(iv) We compute the second derivatives of $A(\theta_1, \theta_2)$, namely

$$\begin{aligned} A_{\theta_1\theta_1} &= \frac{1}{2}(-\sin\theta_2\cos\theta_1 + \sin\theta_1(\cos\theta_2 - 1)), \\ A_{\theta_1\theta_2} &= \frac{1}{2}(-\cos\theta_2\sin\theta_1 + \cos\theta_1\sin\theta_2), \\ A_{\theta_2\theta_2} &= \frac{1}{2}(-\sin\theta_2(\cos\theta_1 - 1) + \sin\theta_1\cos\theta_2). \end{aligned}$$

At the point $(0, 0)$ we have

$$A_{\theta_1\theta_1}(0,0) = 0, \qquad A_{\theta_1\theta_2}(0,0) = 0 \qquad \text{and} \qquad A_{\theta_2\theta_2}(0,0) = 0.$$

The second derivative test is inconclusive, but we know this is a minimum because area is never less than zero.

At the point $(2\pi/3, 4\pi/3)$ we have

$$\begin{aligned} A_{\theta_1\theta_1}(2\pi/3, 4\pi/3) &= -\sqrt{3}/2 \\ A_{\theta_1\theta_2}(2\pi/3, 4\pi/3) &= \sqrt{3}/4 \\ A_{\theta_2\theta_2}(2\pi/3, 4\pi/3) &= -\sqrt{3}/2. \end{aligned}$$

Since

$$\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{3}{4} - \frac{3}{16} > 0,$$

we know that this critical point is a local maximum or a local minimum. Since $-\sqrt{3}/2 < 0$, it is a local maximum, as we expected.

3. (i) We have two lines which are level curves. So both lines are horizontal, parallel to the plane $z = 0$. They meet at a point $P$, so they must live in the same plane $z = c$, for some constant $c$. Now the tangent plane to this point $P$ must contain both lines. Since both lines are horizontal the tangent plane is horizontal. So, $P$ is a critical point.

(ii) $P$ need not be a saddle point. For example, consider the function $f(x, y) = x^2y^2$. We see that $f_x = 2xy^2$ and $f_y = 2x^2y$, so $P = (0, 0)$ is a critical point of $f$.

The level curve

$$f(x, y) = x^2y^2 = 0,$$

is the union of the $x$-axis and the $y$-axis, which intersect at the critical point $(0, 0)$. Moreover, $f(x, y) \geq 0$ for all $x$ and $y$, so $(0, 0)$ is a global minimum of $f$, and so is not a saddle point.