# MODEL ANSWERS TO HWK #3

1. (i) One possible parametrisation of the line passing through the points $E = (2, 0, 0)$ and $P = (x_0, y_0, z_0)$ is

$$\vec{r}(t) = \langle 2, 0, 0 \rangle + t \langle x_0 - 2, y_0, z_0 \rangle = \langle 2 + t(x_0 - 2), ty_0, tz_0 \rangle .$$

A point $Q$ will lie in the $yz$-plane if and only if its $x$-coordinate is equal to 0. Therefore, in order for $Q$ to also lie on the line parametrised by $\vec{r}(t)$, we want the time $t_0$ such that

$$2 + t_0(x_0 - 2) = 0 \Rightarrow t_0 = \frac{2}{2 - x_0},$$

in which case the point $Q$ has coordinates

$$(\frac{2y_0}{2 - x_0}, \frac{2z_0}{2 - x_0}).$$

Note that we can divide through by $2 - x_0$ as we are assuming $x_0 < 2$. This assumption is legitimate because we only need to project something in front of our eyes and since the screen has $x$-coordinate zero, something is in front of us if and only if $x_0 < 2$.

(ii) The image on the screen of a line segment is either a point or a line segment. Suppose the two endpoints of the line segment we are projecting are $A$ and $B$. Let $A_1$ and $B_1$ be the projections of $A$ and $B$ in the $yz$-plane.

There are two cases. If $E$, $A$ and $B$ are collinear then $A_1 = B_1$ and the image of the line segment $AB$ is a single point.

If $E$, $A$ and $B$ are not collinear then $A_1 \neq B_1$ and the image of the line segment $AB$ is the line segment $A_1B_1$.

(iii) From part (i) we have that the projections of $P_0 = (-1, -3, 1)$ and $P_1 = (-2, 4, 6)$ are $Q_0 = (-2, \frac{2}{3})$ and $Q_1 = (2, 3)$ respectively. Since $Q_0 \neq Q_1$ from part (ii) we have that the image of the line segment $P_0P_1$ is the line segment $Q_0Q_1$.

(iv) The parametrisation of the line passing through the points $P_0$ and $P_1$ is given by

$$\vec{r}(t) = \langle -1, -3, 1 \rangle + t \langle -1, 7, 5 \rangle = \langle -1 - t, -3 + 7t, 1 + 5t \rangle .$$

Using part (i) we know that the projection of this parametrisation is

$$\vec{r}_P(t) = \left\langle \frac{2(-3 + 7t)}{2 - (-1 - t)}, \frac{2(1 + 5t)}{2 - (-1 - t)} \right\rangle = \left\langle \frac{14t - 6}{3 + t}, \frac{10t + 2}{3 + t} \right\rangle .$$

Observe that

$$\vec{r}_P(0) = \left\langle -2, \frac{2}{3} \right\rangle = Q_0 \qquad \text{and} \qquad \vec{r_p}(1) = \langle 2, 3 \rangle = Q_1.$$

The line connecting those two points in the $yz$-plane is $12z - 7y = 22$. Note that

$$12\frac{10t+2}{3+t} - 7\frac{14t-6}{3+t} = \frac{22t+66}{3+t} = 22,$$

as expected.
Therefore the projection $\vec{r}_P(t)$ is a line in the $yz$-plane as anticipated in part (ii).
As $t \to \infty$ we have

$$\lim_{t\to\infty} \vec{r}_P(t) = \lim_{t\to\infty} \left\langle \frac{14 - \frac{6}{t}}{\frac{3}{t}+1}, \frac{10 + \frac{2}{t}}{\frac{3}{t}+1} \right\rangle = \langle 14, 10 \rangle .$$

Thus the trajectory on the screen is the line segment joining the points $(-2, \frac{2}{3})$ and $(14, 10)$. The vanishing point is $(0, 14, 10)$.
(v) The top of the fence is a line $l_1$ which passes through the point $(1, 0, 1)$ and is parallel to the $y$-axis. The intersection of the $yz$-plane with the plane, which contains through $E$ and $l_1$, is a line $l_2$ which is also parallel to the $y$-axis. Anything below the line $l_2$ is hidden by the fence.
Part (i) implies that the projection of the point $(1, 0, 1)$ is $(0, 0, 2)$. Since $(0, 0, 2)$ lies on $l_2$ and $l_2$ is parallel to the $y$-axis, $l_2$ is the line $z = 2$ in the $yz$-plane. Hence all points with $z \leq 2$ in the $yz$-plane will not be visible due to the fence.
Part (iv) implies that the trajectory on the screen is a line segment belonging to the line $12z - 7y = 22$. This segment intersects the line $l_2$ at $(\frac{2}{7}, 2)$. Thus the trajectory of the bird that will not be seen is the line segment that joins the points $(-2, \frac{2}{3})$ and $(\frac{2}{7}, 2)$.
2. a) The level curves are squares centered at the origin with sides parallel to the $x$-axis and the $y$-axis.
b) Let $\vec{v} = \langle x, y \rangle$. Recall that the matrix

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

acts on the vector $\vec{v}$ by rotating it $\theta$ radians counterclockwise. If we take $\theta = \frac{\pi}{4}$ then $A_{\frac{\pi}{4}}\vec{v}$ will give us a vector rotated $\pi/4$ radians counterclockwise. Thus the level curves of $f(A_{\frac{\pi}{4}}\vec{v})$ will be the same as the level curves of $f(\vec{v})$ but rotated by an angle of $\pi/4$ radians. Now

$$A_{\pi/4}\vec{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \end{pmatrix}.$$

Therefore, one possible answer to this problem is

$$f(A_{\frac{\pi}{4}}\vec{v}) = f\left(\frac{1}{\sqrt{2}}(x-y), \frac{1}{\sqrt{2}}(x+y)\right) = \max\left(\frac{1}{\sqrt{2}}|x-y|, \frac{1}{\sqrt{2}}|x+y|\right).$$

3. a) The approximation formula is

$$\Delta w = w(x+\Delta x, y+\Delta y) - w(x,y) \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y.$$

When $w = x^y = x^{y \ln x}$, we get

$$\Delta w \approx \frac{y}{x} e^{y \ln x}\Delta x + \ln x e^{y \ln x}\Delta y = x^y \left(\frac{y}{x}\Delta x + \ln x \Delta y\right).$$

When $x = y = 2$, $\Delta x = -0.02$ and $\Delta y = 0.01$, we have

$$\Delta w = w(2-0.02, 2+0.01) - w(2,2) = 1.98^{2.01} - 2^2,$$

whilst the approximation formula gives

$$\Delta w \approx 2^2(1(-0.02) + \ln 2(0.01)) \approx 4(-0.02 + 0.7) = -0.052$$

Using this approximation we get $1.98^{2.01} \approx 4 - 0.052 = 3.948$, which is pretty accurate considering that $1.98^{2.01}$ is 3.9473, to four decimal places.

b) Again using the approximation formula we have

$$\Delta w = w(x+\Delta x, y) - w(x,y) \approx \frac{\partial w}{\partial x}\Delta x = \frac{y}{x}x^y \Delta x$$
$$\Delta w = w(x, y+\Delta y) - w(x,y) \approx \frac{\partial w}{\partial y}\Delta y = \ln x x^y \Delta y.$$

When $x = 2$ and $y = 2$ we obtain

$$w(2+\Delta x, 2) - w(2,2) \approx 4\Delta x$$
$$w(2, 2+\Delta y) - w(2,2) \approx 2.8\Delta y.$$

Therefore, $w$ is more sensitive to small changes in $x$ compared to the exponent $y$.