MODEL ANSWERS TO HWK #2

Part B

1. (a) $M\vec{x}$ is a column vector with four rows. The ith row is the amount of ingredient i contained in x_1 cookies, x_2 doughnuts and x_3 croissants.
(b) $N(M\vec{x}) = (NM)\vec{x}$, a column vector with three rows.
(c) We have $\vec{y} = (NM)\vec{x}$. So $\vec{x} = A\vec{y}$, where $A = (NM)^{-1}$ is the inverse of the product NM. Using the computer algebra package sage, the matrix A to two decimal places is

$$\begin{pmatrix} -0.86 & .08 & .09 \\ 0.64 & -0.01 & -0.17 \\ 0.1 & -0.03 & 0.08 \end{pmatrix}.$$

(d) We want to calculate the product

$$\begin{pmatrix} -0.86 & .08 & .09 \\ 0.64 & -0.01 & -0.17 \\ 0.1 & -0.03 & 0.08 \end{pmatrix} \begin{pmatrix} 50 \\ 300 \\ 65 \end{pmatrix}.$$

Using sage this is

$$\begin{pmatrix} -14.3 \\ 18.2 \\ 1.5 \end{pmatrix},$$

which means we should eat -14.3 cookies, 18.2 doughnuts and 1.5 croissants to get 50 g of protein, 300 g of carbohydrates and 65 g of fat. This doesn't make sense as we cannot eat a negative amount of cookies. The problem is that cookies, doughnuts and croissants just aren't very nutritious and no combination of them has the right amounts of protein, carbohydrates and fat (actually there is another problem; presumably eighteen and one quarter doughnuts contains more than $2,000$ calories; the symptom might be different but the cause is the same).
2. We write out $A\vec{x} = \lambda\vec{x}$ in long form,

$$\begin{aligned} ax + by &= \lambda x \\ cx + dy &= \lambda y. \end{aligned}$$

Putting x and y on one side, we get

$$\begin{aligned} (a - \lambda)x + by &= 0 \\ cx + (d - \lambda)y &= 0. \end{aligned}$$

We can put this back into matrix form as

$$M\vec{x} = \vec{0},$$

where M is the coefficient matrix,

$$M = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}.$$

The advantage of this form is that this is a homogeneous system of linear equations. Since we are assuming that $\vec{x} \neq \vec{0}$, $\det M = 0$ (otherwise the only solution to the homogeneous is $\vec{0}$).
Expanding the determinant, we get

$$0 = \det M = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$$

So,

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is a quadratic equation for λ. A quadratic equation has 2, 1 or 0 solutions. (*In retrospect one can simplify the algebra at the start. Note that $I_2\vec{x} = \vec{x}$. So $\lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$. [$\lambda I$ is the just the diagonol matrix with entries λ on the diagonal.] The matrix equation $A\vec{x} = (\lambda I)\vec{x}$ may be rewritten as $(A - \lambda I)\vec{x} = \vec{0}$ or $M\vec{x} = \vec{0}$, where $M = A - \lambda I$.*)
(b) The discriminant of this quadratic equation is

$$(a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc.$$

There are two solutions is this is positive. There is one solution if this is zero. There are no solutions if this is negative.
So there are two eigenvalues if $(a-d)^2 > -4bc$, one eigenvalue if $(a-d)^2 = -4bc$ and no eigenvalues if $(a-d)^2 < -4bc$.
3. a) Pick the diagonals as in the hint, so that one diagonal connects $(1,0,0)$ to $(1,1,1)$ and the other connects $(1,1,0)$ to $(0,1,1)$. Then the two lines are

$$\vec{r}_1(t) = \langle 1,0,0\rangle + t\langle 0,1,1\rangle \qquad \text{and} \qquad \vec{r}_2(t) = \langle 1,1,0\rangle + t\langle -1,0,1\rangle.$$

There are two ways to find parallel planes containing the lines. First a method which works for any pair of skew lines. A plane parallel to both lines has a normal orthogonal the direction of both lines, $\vec{v} = \langle 0,1,1\rangle$ and $\vec{w} = \langle -1,0,1\rangle$. The cross product is then a normal to the plane

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = \hat{\imath} - \hat{\jmath} + \hat{k}.$$

The plane containing the first line is

$$0 = \langle x-1, y, z\rangle \cdot \langle 1, -1, 1\rangle = (x-1) - y + z,$$

so that $x - y + z = 1$. The plane containing the second line is

$$0 = \langle x-1, y-1, z\rangle \cdot \langle 1, -1, 1\rangle = (x-1) - (y-1) + z,$$

so that $x - y + z = 0$. Pick a point of one plane. Note that $(0, 0, 0)$ belongs to the second plane. The line through this point parallel to $\vec{n}$ intersects the other plane at the closest point. This line is

$$\vec{r}(t) = t\langle 1, -1, 1\rangle,$$

and this belongs to $x - y + z = 1$ when $t = 1/3$. The distance of the point

$$\frac{1}{3}\langle 1, -1, 1\rangle$$

to the origin is $\frac{\sqrt{3}}{3}$.

Alternatively one could use the geometry of the cube to solve this problem. The plane H_1 determined by $(1, 0, 0)$, $(1, 1, 1)$ and $(0, 0, 1)$ contains one diagonal, and the plane H_2 determined by $(1, 1, 0)$, $(0, 1, 1)$ and $(0, 0, 0)$ contains the other diagonal. We have H_1 and H_2 are parallel, and both are perpendicular to the diagonal d connecting $(1, 0, 1)$ and $(0, 1, 0)$. Moreover, the segment of d between H_1 and H_2 is one third of the total length of d, which is $\sqrt{3}/3$.

b) Let $P = (1, s, s)$ and $Q = (1 - t, 1, t)$ be two general points on either line. If P and Q are the closest points then $\overrightarrow{PQ}$ is orthogonal to $\vec{v} = \langle 0, 1, 1\rangle$ and $\vec{w} = \langle -1, 0, 1\rangle$,

$$\overrightarrow{PQ} \cdot \vec{v} = 0 \qquad \text{and} \qquad \overrightarrow{PQ} \cdot \vec{w} = 0.$$

Now

$$\overrightarrow{PQ} = \langle 0, 1, 0\rangle - s\langle 0, 1, 1\rangle + t\langle -1, 0, 1\rangle.$$

So,

$$\begin{aligned} -2s + t &= -1 \\ s + 2t &= 0. \end{aligned}$$

It follows that $t = 1/3$ and $s = 2/3$. The vector connecting the closest points is

$$\langle -1/3, 1/3, -1/3\rangle,$$

which has length $\sqrt{3}/3$.

Alternatively, we want to minimise

$$|\overrightarrow{PQ}| = |(t, s-1, s-t)| = \sqrt{2t^2 + 2s^2 - 2s + 1 - 2st}.$$

Instead of minimising the length, it is easier to minimise the square of the length,

$$f(s,t) = 2s^2 + 2t^2 - 2s - 2st + 1.$$

Take the partials and set them equal to zero,

$$\frac{\partial f}{\partial s} = 4s - 2 - 2t = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial t} = 4t - 2s = 0$$

We get $s = 2/3$ and $t = 1/3$. This is the only critical point and so it must be the minimum.

$$f(2/3, 1/3) = 1/3,$$

so that the distance between these two diagonals is $\sqrt{3}/3$.

3. a) Call the position of the pegs A and B. We have

$$\vec{P} = \frac{1}{2}(\vec{A} + \vec{B}).$$

Call C the centre of the circle of radius 2 and D the centre of the circle of radius 1. t is the angle the peg A has rotated from the y-axis clockwise. So

$$\overrightarrow{CA} = \langle 2\sin t, 2\cos t\rangle.$$

It follows that

$$\vec{A} = \vec{C} + \overrightarrow{CA} = \langle 2\sin t, 2\cos t - 2\rangle.$$

The wheel of radius one is rotating counterclockwise twice as fast as the wheel of radius two. Hence the peg B makes an angle of $2t$ with the $-y$-axis, clockwise. So

$$\overrightarrow{DB} = \langle \sin 2t, -\cos 2t\rangle.$$

It follows that

$$\vec{B} = \vec{D} + \overrightarrow{DB} = \langle \sin 2t, 1 - \cos 2t\rangle.$$

Putting all of this together,

$$\vec{P} = \langle \sin t + \sin t\cos t, \cos t - \frac{1}{2}(1 + \cos 2t)\rangle.$$

b) The velocity vector $\vec{v}(t)$ of $\vec{r}(t)$ is

$$\begin{aligned}\vec{r}'(t) &= \langle (2\cos t + 2\cos 2t)/2, (-2\sin t + 2\sin 2t)/2\rangle \\ &= \langle \cos t + \cos 2t, -\sin t + \sin 2t\rangle.\end{aligned}$$

The acceleration vector $\vec{a}(t)$ of $P(t)$ is

$$\vec{v}'(t) = \langle -\sin t - 2\sin 2t, -\cos t + 2\cos 2t\rangle.$$

So we have $\vec{v}(0) = \langle 2, 0\rangle$ and $\vec{a}(0) = \langle 0, 1\rangle$.