MODEL ANSWERS TO HWK #12

1. (i) The divergence of a gradient field is $\operatorname{div} \nabla f = \nabla \cdot \nabla f = \nabla^2 f$, the Laplacian of f. By the divergence theorem it follows that

$$\oiint_S \nabla f \cdot \mathrm{d}\vec{S} = \iiint_D \nabla^2 f \, \mathrm{d}V.$$

(ii) Again we want to use the divergence theorem. We have

$$\operatorname{div}(f \nabla f) = f_x^2 + f f_{xx} + f_y^2 + f f_{yy} + f_z^2 + f f_{zz} = f \nabla^2 f + |\nabla f|^2.$$

(iii) If f is harmonic then $\nabla^2 f = 0$, so the RHS of (ii), the triple integral, becomes $\iiint_D |\nabla f|^2 \, \mathrm{d}V$. If f is zero on the boundary S of D then the LHS of (ii), the surface integral, is zero. So the triple integral

$$\iiint_D |\nabla f|^2 \, \mathrm{d}V = 0.$$

But we're integrating a non-negative function, which is zero only when ∇f is zero. For the integral to be zero, ∇f must be zero on all of D. It follows that f is constant on D, and since f is zero on the boundary, f must be zero on all of D.
(iv) By part (iii), the function $f - g$ is zero on the solid D, so f and g coincide.
2. (i) The helix C is $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from $t = 0$ to $t = 2\pi$, so

$$\mathrm{d}\vec{r} = \langle -\sin t, \cos t, 1 \rangle \, \mathrm{d}t.$$

and the work is computed by the line integral

$$\begin{aligned}
\oint_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_0^{2\pi} -a \sin^2 t - b \sin^3 t \cos t + 2 \cos^3 t \sin t + \cos^2 t - t^2 \, \mathrm{d}t \\
&= \left[-\frac{at}{2} - \frac{a \cos 2t}{4} - \frac{b \sin^4 t}{4} - \frac{\cos^4 t}{2} + \frac{t}{2} + \frac{\sin 2t}{4} - \frac{t^3}{3} \right]_0^{2\pi} \\
&= (1 - a)\pi - \frac{(2\pi)^3}{3}.
\end{aligned}$$

(ii)

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \langle 0, a \cos z - \cos z, 4xy - 2bxy \rangle.$$

$\vec{F}$ is a gradient field when its curl is zero, so when $a = 1$ and $b = 2$. Since it's defined and continuously differentiable everywhere, $\vec{F}$ is also conservative in this case.
(iii) Suppose f is a potential function. We need to solve three PDE's

$$f_x = \sin z + 2xy^2, \qquad f_y = 2x^2y \qquad \text{and} \qquad f_z = x\cos z - z^2.$$

If we integrate the first PDE with respect to x we get

$$f(x, y, z) = \int \sin z + 2xy^2 \, \mathrm{d}x = x \sin z + x^2y^2 + g(y, z),$$

for some arbitrary function $g(y, z)$ of y and z. If we plug this into the second PDE we get

$$2x^2y + g_y = 2x^2y.$$

Cancelling we get

$$g_y = 0,$$

so that $g(y, z) = h(z)$ is a function only of z. So now we know

$$f(x, y, z) = x \sin z + x^2y^2 + h(z).$$

Plug this into the 3rd PDE, we get

$$x \cos z + h_z = x \cos z - z^2.$$

Cancelling we see that

$$h_z = -z^2.$$

Therefore

$$h(z) = -\frac{z^3}{3},$$

is one possibility for h and so

$$f(x, y, z) = x \sin z + x^2y^2 - \frac{z^3}{3},$$

is one possibility for f.
By the FTC for line integrals we have

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = f(1, 0, 2\pi) - f(1, 0, 0) = -\frac{(2\pi)^3}{3},$$

the same answer as in 2 (i).
3. (i) $P_0P_1P_3$, $P_1P_2P_3$, $P_0P_2P_1$, $P_0P_3P_2$.
(ii) If we parametrise the line segment P_0P_1 by $\vec{r}(t) = \langle t, 0, t\rangle$, $0 \leqslant t \leqslant 1$, then

$$\mathrm{d}\vec{r} = \langle 1, 0, 1\rangle \, \mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle 0, 0, 0\rangle.$$

The work done is then

$$\int_{P_0P_1} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 0\,\mathrm{d}t = 0.$$

If we parametrise the line segment P_1P_3 by $\vec{r}(t) = \langle 1, t, 1-t\rangle$, $0 \leqslant t \leqslant 1$, then

$$\mathrm{d}\vec{r} = \langle 0, 1, -1\rangle\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle t(1-t), 0, -t^2\rangle.$$

The work done is then

$$\int_{P_1P_3} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 t^2\,\mathrm{d}t = 1/3.$$

If we parametrise the line segment P_3P_0 by $\vec{r}(t) = \langle 1-t, 1-t, 0\rangle$, $0 \leqslant t \leqslant 1$, then

$$\mathrm{d}\vec{r} = \langle -1, -1, 0\rangle\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle 0, 0, -(1-t)^2\rangle.$$

The work done is then

$$\int_{P_3P_0} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 0\,\mathrm{d}t = 0.$$

The total work done is therefore

$$\oint_{P_0P_1+P_1P_3+P_3P_0} \vec{F}\cdot \mathrm{d}\vec{r} = \int_{P_0P_1} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{P_1P_3} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{P_3P_0} \vec{F}\cdot \mathrm{d}\vec{r} = \frac{1}{3}.$$

(iii) By Stokes' theorem, the work done going around the boundary of a face is the surface integral of the curl of $\vec{F}$ across that face. The orientations are such that the normal vectors point out of the tetrahedron. Now

$$\begin{aligned} \operatorname{curl}\vec{F} &= \nabla \times \vec{F} \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 0 & -y^2 \end{vmatrix} \\ &= -2y\hat{\imath} + y\hat{\jmath} - y^2\hat{k} \\ &= \langle -2y, y, -z\rangle. \end{aligned}$$

and by Hwk #11, Part B 4, the normal vectors to the faces S_0, S_1, S_2 and S_3 are

$$\vec{n}_0 = \hat{\imath}, \quad \vec{n}_1 = -\hat{\imath} + \hat{\jmath} - \hat{k}, \quad \vec{n}_2 = \hat{\imath} - \hat{\jmath} + \hat{k} \quad \text{and} \qquad \vec{n}_3 = -\hat{\jmath}.$$

The flux of curl $\vec{F} = \langle -2y, y, -z\rangle$ and the flux of $\vec{G} = \langle -2y, 0, 0\rangle$ across the face S_0 are therefore the same, as both dot $\vec{n}_0 = \hat{\imath}$ the same and so

$$\iint_{S_0} \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S_0} \vec{G} \cdot \mathrm{d}\vec{S} = -\frac{2}{3},$$

as calculated in Hwk #11, Part B 4. The face S_3 is contained in the plane $y = 0$. Hence curl $F = \langle 0, 0, -z\rangle$ along S_3, so that curl $\vec{F} \cdot \hat{n}_3 = 0$. The flux across S_3 is therefore zero.
Finally recall that $z \longrightarrow -z$ switches S_1 and S_2. Since it flips the sign of $\hat{k}$ in both the normal vector and $\vec{F}$, the flux across these sides is equal.
So we only need to calculate what happens across S_1. Let's project onto the xy-plane. As $|\vec{n}_1 \cdot \hat{k}| = 1$, we have

$$\mathrm{d}\vec{S} = \langle -1, 1, -1\rangle \, \mathrm{d}x \, \mathrm{d}y.$$

The image of S_1 is the triangle R with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. The equation of the plane containing the face S_1 is $x - y + z = 0$ (it contains the origin P_0 and $\vec{n}_1$ is a normal vector). Thus $z = y - x$ on the S_1 and the flux is

$$\begin{aligned}\int_{S_1} \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} &= \int_R \langle -2y, y, x - y\rangle \cdot \langle -1, 1, -1\rangle \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_R 4y - x \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \int_0^x 4y - x \, \mathrm{d}y \, \mathrm{d}x.\end{aligned}$$

The inner integral is

$$\int_0^x 4y - x \, \mathrm{d}y = \Big[2y^2 - xy \Big]_0^x = x^2.$$

The outer integral is

$$\int_0^1 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

So the work done across either S_1 or S_2 is 1/3.
(iv) First we note that the sum of the surface integrals is indeed zero:

$$\int_{S_0} \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} + \int_{S_1} \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} + \int_{S_2} \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} + \int_{S_3} \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = \frac{-2}{3} + \frac{1}{3} + \frac{1}{3} + 0 = 0.$$

(a) By Stokes' theorem, the sum of the four surface integrals equals the sum over all the boundary curves, in which every edge will be included twice, but with the opposite orientation. In other words, every edge is part of the boundary of two faces, but the orientation it is given when considered as the edge of one face is the opposite of the other orientation. So the sum of the 8 line integrals is zero, since we can pair them into 4 sets of two line integrals whose sum is zero.

(b) By the divergence theorem, the sum of the surface integrals equals the integral over the solid tetrahedron of

$$\operatorname{div}\operatorname{curl}\vec{F} = \operatorname{div}\langle -2y, y, -z\rangle = 1 - 1 = 0.$$