MODEL ANSWERS TO HWK #11

Note there are some mistake in the back of the book. (12.8.55.b), that should be cosecant not secant and the range of ϕ is given by $\pi/6 \leq \phi \leq 5\pi/6$.
There is also a mistake in the solutions to 6B-8. The range for θ should be $0 \leq \theta \leq \pi$.
1. Let B be the ball given by

$$x^2 + y^2 + (z-a)^2 \leq a^2.$$

The average distance is:

$$\bar{\rho} = \frac{1}{\mathrm{vol}(B)} \iiint_B \rho \, \mathrm{d}V = \frac{3}{4\pi a^3} \iiint_B \sqrt{x^2+y^2+z^2} \, \mathrm{d}V.$$

First we work in cylindrical coordinates, so that $r^2 = x^2 + y^2$.

$$\iiint_B \rho \, \mathrm{d}V = \int_0^{2a} \int_0^{2\pi} \int_0^{\sqrt{a^2-(z-a)^2}} r\sqrt{r^2+z^2} \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z.$$

The inner integal is

$$\begin{aligned} \int_0^{\sqrt{a^2-(z-a)^2}} r\sqrt{r^2+z^2} \, \mathrm{d}r &= \left[\frac{1}{3}(r^2+z^2)^{3/2}\right]_0^{\sqrt{2az-z^2}} \\ &= \frac{1}{3}((2az)^{3/2} - z^3). \end{aligned}$$

The middle integral is

$$\frac{1}{3}\int_0^{2\pi} (2az)^{3/2} - z^3 \, \mathrm{d}\theta = \frac{2\pi}{3}((2az)^{3/2} - z^3).$$

The outer integral is

$$\begin{aligned} \frac{2\pi}{3}\int_0^{2a} (2az)^{3/2} - z^3 \, \mathrm{d}z &= \frac{2\pi}{3}\left[\frac{2}{5}(2a)^{3/2}z^{5/2} - \frac{1}{4}z^4\right]_0^{2a} \\ &= \frac{2\pi}{3}(\frac{2}{5}(2a)^5 - \frac{1}{4}(2a)^4) \\ &= \frac{8\pi a^4}{5}. \end{aligned}$$

So

$$\bar{\rho} = \frac{1}{\mathrm{vol}(B)} \iiint_B \rho \, \mathrm{d}V = \frac{6a}{5}.$$

Now let's use spherical coordinates. We calculate the equation for ρ in terms of ϕ. Cutting by the plane $y = 0$, we get a circle and we are down to a calculation in the plane. There are two ways to proceed. We could use a classic piece of geometry; the angle subtended by a diameter on the circumference of a circle is always a right angle.

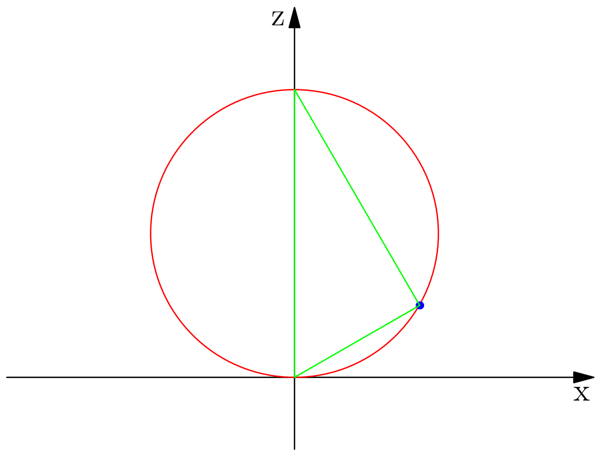

FIGURE 1. Classic geometry

The hypotenuse of this triangle is $2a$ and ρ is the adjacent side to the angle ϕ, so

$$\rho = 2a\cos\phi.$$

Or we can proceed as in lecture 16. The equation for the circle in Cartesian coordinates is

$$x^2 + (y - a)^2 = a^2.$$

Expanding and simplifying, we get

$$x^2 + y^2 = 2ay.$$

But $\rho^2 = x^2 + y^2$ and $y = \rho\cos\phi$, so that

$$\rho = 2a\cos\phi,$$

as before.
Thus

$$\iiint_B \rho \, \mathrm{d}V = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2a\cos\phi} \rho^3 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

The inner integral is

$$\int_0^{2a\cos\phi} \rho^3 \sin\phi \,\mathrm{d}\rho = \left[\frac{1}{4}\rho^4 \sin\phi\right]_0^{2a\cos\phi}$$
$$= 4a^4 \sin\phi \cos^4\phi.$$

The middle integral is

$$4a^4 \int_0^{2\pi} \sin\phi \cos^4\phi \,\mathrm{d}\theta \,\mathrm{d}\phi = 8\pi a^4 \sin\phi \cos^4\phi.$$

The outer integral is

$$8\pi a^4 \int_0^{\pi/2} \sin\phi \cos^4\phi \,\mathrm{d}\phi = 8\pi a^4 \left[-\frac{1}{5}\cos^5\phi\right]_0^{\pi/2}$$
$$= \frac{8\pi a^4}{5},$$

so we get the same answer as in cylindrical coordinates.

2. Let us place the sphere as in part 1. Using symmetry we see that only the z component of the gravitational force is non-zero. Let B be the hemisphere. Let us compute using spherical coordinates. We have

$$F_z = G\iiint_B \frac{\cos\phi}{\rho^2}\,\mathrm{d}V = \iiint_B \cos\phi \sin\phi \,\mathrm{d}\rho \,\mathrm{d}\theta \,\mathrm{d}\phi.$$

The sphere is symmetric in θ. Cutting by the plane $y = 0$ we get a circle.

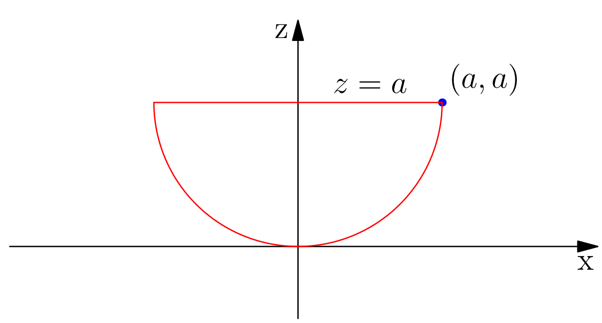

FIGURE 2. Cross section of hemisphere

Equivalently the solid of revolution obtained by rotating a quarter circle is a hemisphere. We have to divide the integral into two pieces. For $0 \leq \phi \leq \pi/4$, ρ is bounded above by the line $z = a$.

In this case ρ is the hypotenuse of a right-angled triangle with side a and angle ϕ, $\rho = a\csc(\phi)$.

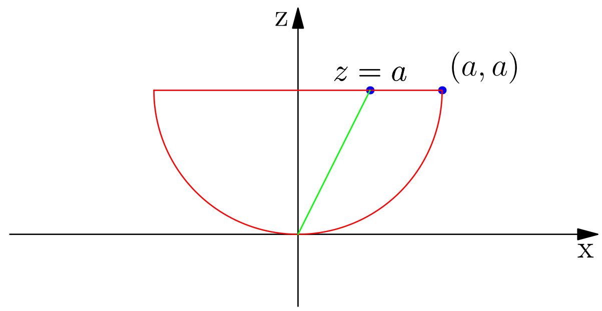

FIGURE 3. $\phi \leq \pi/4$

For $\pi/4 \leq \phi \leq \pi/2$, ρ is bounded above by the circle and we already figured out the limits in this case. So we have to calculate:

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{a\sec\phi}\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi+\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{2a\cos\phi}\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

We compute the first integral. The inner integral is

$$\int_0^{a\sec\phi}\cos\phi\sin\phi\,\mathrm{d}\rho=\Big[\cos\phi\sin\phi\rho\Big]_0^{a\sec\phi}=a\sin\phi.$$

The middle integral is

$$\int_0^{2\pi}a\sin\phi\,\mathrm{d}\theta=2\pi a\sin\phi.$$

The outer integral is

$$2\pi a\int_0^{\pi/4}\sin\phi\,\mathrm{d}\phi=2\pi a\Big[-\cos\phi\Big]_0^{\pi/4}=2\pi a(1-\frac{1}{\sqrt{2}}).$$

We compute the second integral. The inner integal is

$$\int_0^{2a\cos\phi}\cos\phi\sin\phi\,\mathrm{d}\rho=\Big[\cos\phi\sin\phi\rho\Big]_0^{2a\cos\phi}=2a\sin\phi\cos^2\phi.$$

The middle integral is

$$\int_0^{2\pi}2a\sin\phi\cos^2\phi\,\mathrm{d}\theta=4\pi a\sin\phi\cos^2\phi$$

The outer integral is

$$4\pi a\int_{\pi/4}^{\pi/2}\sin\phi\,\mathrm{d}\phi=4\pi a\Big[-\frac{1}{3}\cos^3\phi\Big]_{\pi/4}^{\pi/2}=\frac{\sqrt{2}}{3}\pi a.$$

Therefore

$$F_z = 2\pi Ga(1 - \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}}) = 2\pi Ga(1 - \frac{\sqrt{2}}{3}).$$

We could also compute this using cylindrical coordinates.

$$F_z = G \int_0^a \int_0^{2\pi} \int_0^{\sqrt{a^2-(z-a)^2}} \frac{rz}{(r^2+z^2)^{3/2}} \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z.$$

The inner integal is

$$\begin{aligned} \int_0^{\sqrt{a^2-(z-a)^2}} rz(r^2+z^2)^{-3/2} \, \mathrm{d}r &= \Big[-z(r^2+z^2)^{-1/2} \Big]_0^{\sqrt{2az-z^2}} \\ &= 1 - \sqrt{\frac{1}{2a}} z^{1/2} \end{aligned}$$

The middle integral is

$$\int_0^{2\pi} 1 - \sqrt{\frac{1}{2a}} z^{1/2} \, \mathrm{d}\theta = 2\pi \left(1 - \sqrt{\frac{1}{2a}} z^{1/2} \right).$$

The outer integral is

$$\begin{aligned} 2\pi \int_0^a 1 - \sqrt{\frac{1}{2a}} z^{1/2} \, \mathrm{d}z &= 2\pi \left[z - \sqrt{\frac{1}{2a}} \frac{2}{3} z^{3/2} \right]_0^a \\ &= 2\pi a \left(1 - \frac{\sqrt{2}}{3} \right). \end{aligned}$$

Therefore

$$F_z = 2\pi Ga(1 - \frac{\sqrt{2}}{3}).$$

3. Parameterise S using θ and z so that $x = \cos\theta$, $y = \sin\theta$, $z = z$ and so

$$\mathrm{d}\vec{S} = \langle x, y, 0 \rangle \, \mathrm{d}z \, \mathrm{d}\theta = \langle \cos\theta, \sin\theta, 0 \rangle \, \mathrm{d}z \, \mathrm{d}\theta.$$

The integral is therefore

$$\begin{aligned}\iint_S \vec{F}\cdot \mathrm{d}\vec{S} &= \int_{-\infty}^{\infty}\int_0^{2\pi} \frac{1}{1+z^2}\langle \cos\theta, \sin\theta, z\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle \,\mathrm{d}\theta\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_0^{2\pi} \frac{1}{1+z^2}\,\mathrm{d}\theta\,\mathrm{d}z \\ &= 2\pi \int_{-\infty}^{\infty} \frac{1}{1+z^2}\,\mathrm{d}z \\ &= 2\pi \Big[\arctan z \Big]_{-\infty}^{\infty} \\ &= 2\pi^2.\end{aligned}$$

4 (i) $P_0 = (0,0,0)$, $P_1 = (1,0,1)$, $P_2 = (1,0,-1)$ and $P_3 = (1,1,0)$. Note that P_0 and P_3 are fixed by $z \longrightarrow -z$ and P_1 and P_2 are exchanged by $z \longrightarrow -z$. So the faces $S_2 = P_0P_1P_3$ and $S_1 = P_0P_2P_3$ are exchanged by $z \longrightarrow -z$.
(ii) Note that $\overrightarrow{P_1P_2} = -2\hat{k}$, $\overrightarrow{P_1P_3} = \hat{\jmath} - \hat{k}$ are two vectors in the face $S_0 = P_1P_2P_3$ and so the cross-product

$$\vec{n}_0 = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & -2 \\ 0 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 2\hat{\imath},$$

is normal to the face S_0. $P_0 = (0,0,0)$ is a point of the tetrahedron not on S_0. As it has smaller x-coordinate, $\vec{n}_0$ points outwards.
Note that we can check the answer quickly. As the face is fixed by the map $z \longrightarrow -z$, the outward normal is also fixed by the map $z \longrightarrow -z$, which is true of $2\hat{\imath}$. Visibly this vector is orthogonal to $-2\hat{k}$ and $\hat{\jmath} - \hat{k}$.
$\overrightarrow{P_0P_1} = \hat{\imath} + \hat{k}$, $\overrightarrow{P_0P_2} = \hat{\imath} - \hat{k}$ and so

$$\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 2\hat{\jmath}$$

is normal to the face $S_3 = P_0P_1P_2$. P_3 belongs to the tetrahedron but not to this face and P_3 has larger y-coordinate, so $\vec{n}_3 = -2\hat{\jmath}$ is an outwards normal.
We do a quick check, as before. Since this face is fixed by $z \longrightarrow -z$, the outwards normal has no component in the direction of $\hat{k}$ and visibly $\vec{n}_3$ is orthogonal to $\hat{\imath} + \hat{k}$ and $\hat{\imath} - \hat{k}$.

$\overrightarrow{P_0P_2} = \hat{\imath} - \hat{k}$, $\overrightarrow{P_0P_3} = \hat{\imath} + \hat{\jmath}$ and so

$$\vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = \hat{\imath} - \hat{\jmath} + \hat{k}.$$

is normal to the face $S_1 = P_0P_2P_3$. As $\vec{u} \cdot \overrightarrow{P_0P_1} > 0$ and $\overrightarrow{P_0P_1}$ points into the tetrahedron, so does $\vec{u}$. Thus $\vec{n}_1 = -\vec{u} = \langle -1, 1, -1 \rangle$ points outwards.

As $z \longrightarrow -z$ exchanges S_1 and S_2, it follows that $\vec{n}_2 = \langle -1, 1, 1 \rangle$, the vector we get by flipping the sign of the last coordinate of $\vec{n}_1$, is an outward normal to the face S_2.

(iii) As the face S_3 is contained in the plane $y = 0$, $\vec{F} = \vec{0}$ on S_3 and so

$$\iint_{S_3} \vec{F} \cdot \mathrm{d}\vec{S} = 0.$$

For S_0, $\hat{n}_0 = \hat{\imath}$ and so the natural thing to do is project S_0 onto the yz-plane. The shadow of S_0 is the triangle R with vertices $(0, 1)$, $(0, -1)$ and $(1, 0)$. The flux is then the integral of $-y$ over the triangle:

$$\iint_{S_0} \vec{F} \cdot \mathrm{d}\vec{S} = -\iint_R y \, \mathrm{d}A = -\int_0^1 \int_{-1+y}^{1-y} y \, \mathrm{d}z \, \mathrm{d}y.$$

The inner integal is

$$\int_{-1+y}^{1-y} y \, \mathrm{d}z = \Big[yz \Big]_{-1+y}^{1-y} = 2y(1 - y).$$

So the flux is

$$\int_0^1 2y^2 - 2y \, \mathrm{d}y = \left[\frac{2}{3} y^3 - y^2 \right]_0^1 = -\frac{1}{3}.$$

For S_1, let us project onto the xy-plane. The shadow of S_1 is the triangle R with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. We have

$$\mathrm{d}\vec{S} = \frac{\vec{n}_1}{|\vec{n}_1 \cdot \hat{k}|} \, \mathrm{d}x \, \mathrm{d}y = \vec{n}_1 \, \mathrm{d}x \, \mathrm{d}y,$$

and so the function we want to integrate is

$$\vec{F} \cdot \vec{n}_1 = y,$$

that is

$$\iint_{S_1} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_R y \, \mathrm{d}A = \int_0^1 \int_0^x y \, \mathrm{d}y \, \mathrm{d}x.$$

The inner integal is

$$\int_0^x y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_0^x = \frac{x^2}{2}.$$

So the flux is

$$\frac{1}{2}\int_0^1 x^2\,\mathrm{d}z = \frac{1}{6}.$$

The flux out of S_2 is the same, by symmetry, since the vector field $\vec{F}$ is fixed by the symmetry $z \longrightarrow -z$.

(iv) Let S be the surface of the tetrahedron. Then $S = S_0+S_1+S_2+S_3$, the sum of the four faces. Therefore

$$\begin{aligned}\iint_S \vec{F}\cdot\mathrm{d}\vec{S} &= \iint_{S_0} \vec{F}\cdot\mathrm{d}\vec{S} + \iint_{S_1} \vec{F}\cdot\mathrm{d}\vec{S} + \iint_{S_2} \vec{F}\cdot\mathrm{d}\vec{S} + \iint_{S_3} \vec{F}\cdot\mathrm{d}\vec{S}\\ &= -\frac{1}{3}+\frac{1}{6}+\frac{1}{6}+0 = 0,\end{aligned}$$

by our answer to (iii). On the other hand the divergence of $\vec{F}$ is zero, so the RHS of the equation in the divergence theorem is also zero.

5 (i) Note that

$$\rho_x = \frac{x}{\rho}, \qquad \rho_y = \frac{y}{\rho} \qquad \text{and} \qquad \rho_z = \frac{z}{\rho}.$$

Therefore, using the chain rule, we have

$$\begin{aligned}\vec{F}(x,y,z) &= \nabla f(x,y,z)\\ &= -\frac{1}{\rho^2}\left(\frac{x}{\rho}\hat{\imath} + \frac{y}{\rho}\hat{\jmath} + \frac{z}{\rho}\hat{k}\right)\\ &= -\frac{1}{\rho^3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).\end{aligned}$$

This vector field points to the origin and its magnitude is the inverse of the square of the distance to the origin. This is the prototype of a gravitational or an electrical force field.

(ii) Let S be the sphere of radius a centred at the origin. At any point of S the direction of the outward unit normal $\hat{n}$ is opposite to $\vec{F}$ and so

$$\vec{F}\cdot\hat{n} = -|\vec{F}| = -\frac{1}{a^2}.$$

Therefore

$$\begin{aligned}\iint_S \vec{F} \cdot \mathrm{d}\vec{S} &= \iint_S \vec{F} \cdot \hat{n} \,\mathrm{d}A \\ &= \iint_S (-\frac{1}{a^2}) \,\mathrm{d}A \\ &= -\frac{1}{a^2} \iint_S \mathrm{d}A \\ &= -\frac{1}{a^2} \,\mathrm{area}(S) \\ &= -\frac{1}{a^2} \cdot 4\pi a^2 \\ &= -4\pi.\end{aligned}$$

(iii) If we set

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k},$$

then

$$P = -\frac{x}{\rho^3},$$

so that

$$P_x = -\frac{\rho^3 - 3x\rho^2\frac{x}{\rho}}{\rho^6} = \frac{3x^2 - \rho^2}{\rho^5}.$$

Therefore

$$\begin{aligned}\operatorname{div} \vec{F} &= P_x + Q_y + R_z \\ &= \frac{3x^2 - \rho^2 + 3y^2 - \rho^2 + 3z^2 - \rho^2}{\rho^5} \\ &= 3\frac{x^2 + y^2 + z^2 - \rho^2}{\rho^5} \\ &= 0.\end{aligned}$$

This is consistent with the divergence theorem (how could it be otherwise, since the divergence theorem is true?).

The field $\vec{F}$ is defined everywhere but the origin. So the divergence theorem applies to any closed surface which does not enclose the origin; for such surfaces the divergence theorem implies the flux is zero. However the surface S does enclose the origin, since S is a sphere centred at the origin. So we cannot apply the divergence theorem directly to S.

In fact the divergence theorem, applied to a region R between two surfaces, implies that the flux through a closed surface S which encloses the origin is independent of the surface. This is compatible with the answer to part (ii); whatever the radius of S the integral comes out the same (in this case, apply the divergence theorem to the region between two concentric spheres).