MODEL ANSWERS TO HWK #10

1. False. Consider the regions

$$R_1 = \{ 1 < r < 2, 0 \leq \theta < \frac{\pi}{4} \text{ or } \frac{\pi}{2} \leq \theta < 2\pi \},$$
$$R_2 = \{ 1 < r < 2, \frac{\pi}{4} \leq \theta < 2\pi \}.$$

Both of these are portions of an annulus and so they are simply connected. However the union is the whole annulus

$$R_1 \cup R_2 = \{ 1 < r < 2, 0 \leq \theta < 2\pi \},$$

which is not simply connected.

2. Let

$$\vec{F} = \langle M, N \rangle = \langle \frac{x}{r^2}, \frac{y}{r^2} \rangle$$

and let C be a circle of radius a centered at $(1, 0)$. Recall that

$$r_x = \frac{x}{r} \qquad \text{and} \qquad r_y = \frac{y}{r}.$$

Therefore

$$M_x = \frac{r^2 - 2x^2}{r^4} \qquad \text{and} \qquad N_y = \frac{r^2 - 2y^2}{r^4}.$$

It follows that the divergence

$$\operatorname{div} \vec{F} = \frac{r^2 - 2x^2 + r^2 - 2y^2}{r^4} = 0.$$

If $a < 1$ then C is contained in the right half plane and so $\vec{F}$ is defined and differentiable on the whole of the region R bounded by C. The normal form of Green's theorem implies that

$$\oint_C \vec{F} \cdot \hat{n} \, \mathrm{d}s = \iint_R \operatorname{div} \vec{F} \, \mathrm{d}A = 0.$$

Suppose that $a > 1$. Then the region bounded by C contains the origin, a point where $\vec{F}$ is not defined. Let D be the circle centered at the origin with radius $2 + a$. Pick any line segment L connecting D to C (e.g take L along the x-axis). Let $C' = D + L - C - L$ be the closed curve, which first traces out D, then traces L from D to C, goes around C clockwise and then goes back along the same line segment. Let R be the region contained in D but not in C. Then driving along C' we always have the region R on our left. Also R does not contain the origin, so that $\vec{F}$ is defined on the whole of R. Green's theorem implies that

$$\oint_{C'} \vec{F}\cdot\hat{n}\,\mathrm{d}s = \iint_R \operatorname{div}\vec{F}\,\mathrm{d}A = 0.$$

It follows that

$$\begin{aligned}
0 &= \oint_{C'} \vec{F}\cdot\hat{n}\,\mathrm{d}s \\
&= \oint_{D+L-C-L} \vec{F}\cdot\hat{n}\,\mathrm{d}s \\
&= \oint_{D} \vec{F}\cdot\hat{n}\,\mathrm{d}s + \oint_{L} \vec{F}\cdot\hat{n}\,\mathrm{d}s - \oint_{C} \vec{F}\cdot\hat{n}\,\mathrm{d}s - \oint_{L} \vec{F}\cdot\hat{n}\,\mathrm{d}s \\
&= \oint_{D} \vec{F}\cdot\hat{n}\,\mathrm{d}s - \oint_{C} \vec{F}\cdot\hat{n}\,\mathrm{d}s.
\end{aligned}$$

It follows that

$$\oint_{C} \vec{F}\cdot\hat{n}\,\mathrm{d}s = \oint_{D} \vec{F}\cdot\hat{n}\,\mathrm{d}s.$$

Repeating the same argument, but now with a curve of radius one about the origin, we see that if $a > 1$ the flux of $\vec{F}$ across C is the same as the flux of $\vec{F}$ across the unit circle. We calculate this directly. Note that $\vec{F}$ points in the direction of the normal vector. So

$$\vec{F}\cdot\hat{n} = |\vec{F}| = 1,$$

on the unit circle. Thus the flux is the length of the unit circle 2π. Putting all of this together we see that

$$\oint_{C} \vec{F}\cdot\hat{n}\,\mathrm{d}s = 2\pi,$$

if $a > 1$.

Here is another way to proceed. Let

$$\vec{G} = \langle -\frac{y}{r^2}, \frac{x}{r^2}\rangle.$$

Then $\vec{G}$ is the vector field $\vec{F}$ rotated through $\pi/2$ radians, counterclockwise. It follows that to calculate the flux across C for $\vec{F}$ is the same as to calculate the work done along C for $\vec{G}$. $\vec{G}$ was analysed in Hwk # 8, 5. Recall that $\vec{G}$ is not defined at $(0, 0)$. Also $\operatorname{curl}\vec{G} = 0$ by Hwk # 9, 1 (iv). If $a < 1$ then C is contained in the right half plane. In Hwk #8, 5 (i) we showed that there is a potential function for $\vec{G}$ in the right half plane, thus for $a < 1$

$$\oint_{C} \vec{G}\cdot\mathrm{d}\vec{r} = 0.$$

If $a > 1$ then C contains the point $(0, 0)$ in its interior. We already saw in class that if $a > 1$ the work done is 2π. It is essentially a repeat of the argument above, with $\vec{G}$ replacing $\vec{F}$ and the usual form of Green's theorem replacing the normal form.

3. We need to calculate several triple integrals to find the centroid.

$$\iiint_R \mathrm{d}V, \qquad \iiint_R y\,\mathrm{d}V, \qquad \text{and} \qquad \iiint_R z\,\mathrm{d}V$$

will be sufficient by symmetry.

We can solve the first with rectangular coordinates

$$\iiint_R \mathrm{d}V = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

The inner integral is

$$\int_0^{\sqrt{1-z^2}} \mathrm{d}x = \Big[x\Big]_0^{\sqrt{1-z^2}} = \sqrt{1-z^2}.$$

The middle integral is

$$\int_0^{\sqrt{1-z^2}} \sqrt{1-z^2}\,\mathrm{d}y = \Big[y\sqrt{1-z^2}\Big]_0^{\sqrt{1-z^2}} = 1 - z^2.$$

So the outer integral is

$$\int_0^1 1 - z^2\,\mathrm{d}z = \left[z - \frac{z^3}{3}\right]_0^1 = \frac{2}{3}.$$

For the second integral we use cylindrical coordinates where we use polar in the xz-plane

$$\iiint_R y\,\mathrm{d}V = \int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{1-r^2\sin^2\theta}} yr\,\mathrm{d}y\mathrm{d}r\mathrm{d}\theta.$$

The inner integral is

$$\int_0^{\sqrt{1-r^2\sin^2\theta}} yr\,\mathrm{d}y = \left[\frac{1}{2}y^2 r\right]_0^{\sqrt{1-r^2\sin^2\theta}} = \frac{1}{2}(r - r^3\sin^2\theta).$$

The middle integral is

$$\frac{1}{2}\int_0^1 r - r^3\sin^2\theta\,\mathrm{d}r = \frac{1}{2}\left[\frac{r^2}{2} - \frac{r^4}{4}\sin^2\theta\right]_0^1 = \frac{1}{4} - \frac{1}{8}\sin^2\theta.$$

The outer integral is

$$\frac{1}{16}\int_0^{\pi/2} 4 - 2\sin^2\theta\,\mathrm{d}\theta = \frac{1}{16}\left[3\theta + \frac{1}{2}\sin(2t)\right]_0^{\pi/2} = \frac{3\pi}{32}.$$

For the third integral we use rectangular coordinates

$$\iiint_R z\,\mathrm{d}V = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

The inner integral is

$$\int_0^{\sqrt{1-z^2}} z\,\mathrm{d}x = \Big[zx\Big]_0^{\sqrt{1-z^2}} = z\sqrt{1-z^2}.$$

The middle integral is

$$\int_0^{\sqrt{1-z^2}} z\sqrt{1-z^2}\,\mathrm{d}y = \Big[yz\sqrt{1-z^2}\Big]_0^{\sqrt{1-z^2}} = z - z^3.$$

So the outer integral is

$$\int_0^1 z - z^3\,\mathrm{d}z = \left[\frac{z^2}{2} - \frac{z^4}{4}\right]_0^1 = \frac{1}{4}.$$

Thus we conclude that the centroid is $(9\pi/64, 9\pi/64, 3/8)$.