MODEL ANSWERS TO HWK #1

Part B

1. (a) The four vertices are $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ and $(-1, -1, 1)$. The distance between the first two vertices is $2\sqrt{2}$, since two coordinates differ by 2. There are six edges, corresponding to the choice of six pairs of vertices. For each pair of vertices, exactly two coordinates have a different sign, so the length of any edge is $2\sqrt{2}$. Or one could say that every edge is a diagonal of a face, so that all sides have the same length.
(b) Imagine putting hydrogen atoms at the vertices of the tetrahedron from (a). We want to put the carbon atom at the centre of the tetrahedron. This is given by the vector

$$\frac{1}{4}\left(\langle 1, 1, 1\rangle + \langle 1, -1, -1\rangle + \langle -1, 1, -1\rangle + \langle -1, -1, 1\rangle\right) = \langle 0, 0, 0\rangle.$$

In other words, the carbon atom goes at the origin. Take the two hydrogen atoms $(1, 1, 1)$ and $(1, -1, -1)$. Let θ be the angle between the two vectors $\langle 1, 1, 1\rangle$ and $\langle 1, -1, -1\rangle$. We have

$$\cos\theta = \frac{\langle 1, 1, 1\rangle \cdot \langle 1, -1, -1\rangle}{|\langle 1, 1, 1\rangle||\langle 1, -1, -1\rangle|} = -\frac{1}{3}.$$

So the bond angle $\theta \approx 1.91$ radians or 109.47 degrees.
(c) Find the angle at $(1, 1, 1)$ made by the two points $(1, -1, -1)$ and $(-1, 1, -1)$. Let

$$\vec{u} = \langle 0, -2, -2\rangle \qquad \text{and} \qquad \vec{v} = \langle -2, 0, -2\rangle.$$

We want the angle α between $\vec{u}$ and $\vec{v}$. Same as the angle α between $\vec{u}/2$ and $\vec{v}/2$.

$$\cos\alpha = \frac{\langle 0, -1, -1\rangle \cdot \langle -1, 0, -1\rangle}{|\langle 0, -1, -1\rangle||\langle -1, 0, -1\rangle|} = \frac{1}{2}.$$

So the angle is $\pi/3$. Since we have a regular tetrahedron, the faces are regular polygons. In other words the faces are equilateral triangles and so the angle is $\pi/3$.
Find the angle β between the vectors $\vec{u}$ and $\vec{w}$ given by the two line segments $(1, 1, 1)$, $(1, -1, -1)$ and $(-1, 1, -1)$, $(-1, -1, 1)$.

$$\vec{u} = \langle 0, -2, -2\rangle \qquad \text{and} \qquad \vec{w} = \langle 0, -2, 2\rangle.$$

Since $\vec{u} \cdot \vec{w} = 0$, these two vectors are orthogonal. So $\beta = \pi/2$.
Imagine if we orient the tetrahedron so that one edge is at the top so that the opposite edge is at the bottom. If we squash the tetrahedron so it becomes flat, then we want the angle between the top line and the bottom line. The top of the tetrahedron has become two triangles and the top edge is a common edge of both triangles. The bottom edge divides both triangles into equal halves; it is obvious that then the angle is $\pi/2$.
Here is another way to see the angle is $\pi/2$. Suppose we took the first two vertices in the opposite order. The angle wouldn't change and neither would $\vec{w}$, but $\vec{u}$ changes sign. So the sign of $\vec{u} \cdot \vec{w}$ would change. Since the angle is unchanged, and the lengths of $\vec{u}$ and $\vec{w}$ are unchanged, we must have $\vec{u} \cdot \vec{w} = 0$.
There are other ways to argue by symmetry that the angle is $\pi/2$.
(d) We already decided in (c) that the vectors

$$\vec{u} = \langle 0, -2, -2 \rangle \qquad \text{and} \qquad \vec{v} = \langle -2, 0, -2 \rangle,$$

are vectors giving two sides of a face of the tetrahedron. The magnitude of the cross product of $\vec{u}$ and $\vec{v}$ is then twice the area of a face.

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -2 & -2 \\ -2 & 0 & -2 \end{vmatrix} \\ &= \hat{\imath} \begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} \\ &= 4\hat{\imath} + 4\hat{\jmath} - 4\hat{k}. \end{aligned}$$

The length of this vector is $4\sqrt{3}$. So the area of a face is $2\sqrt{3}$.
2. (a) We want to know if the vectors $\vec{v}_1$ and $\overrightarrow{P_2P_3}$ are orthogonal. This happens if and only if $\vec{v}_1 \cdot \overrightarrow{P_2P_3} = 0$. But $\overrightarrow{P_2P_3} = \vec{v}_3 - \vec{v}_2$. So, we want to know if $\vec{v}_1 \cdot (\vec{v}_3 - \vec{v}_2) = 0$. In other words P lies on the altitude of the triangle $P_1P_2P_3$ from the vertex P_1 if and only if

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3.$$

(b) We already decided that if P lies on the altitude from P_1, then

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3.$$

Similarly, if P lies on the altitudes from P_2, then

$$\vec{v}_2 \cdot \vec{v}_1 = \vec{v}_2 \cdot \vec{v}_3.$$

As $\vec{v}_2 \cdot \vec{v}_1 = \vec{v}_1 \cdot \vec{v}_2$, we have

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3.$$

(c) We have

$$\vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3.$$

So

$$\vec{v}_3 \cdot \vec{v}_1 = \vec{v}_3 \cdot \vec{v}_2,$$

and this happens only if P lies on the altitude from P_3.
3. Pick a vertex P of the tetrahedron and let $\vec{u}$, $\vec{v}$, $\vec{w}$ be the three vectors starting at P which end at the three other vertices of the tetrahedron. Suppose that $\vec{u}$, $\vec{v}$ and the outward normal to this face form a right handed set (if they don't then simply switch $\vec{u}$ and $\vec{v}$). Then this is true for $\vec{v}$, $\vec{w}$ and $\vec{w}$, $\vec{u}$. Then three of the vectors we want are the three cross products $\vec{n}_1 = 1/2\vec{u} \times \vec{v}$, $\vec{n}_2 = 1/2\vec{v} \times \vec{w}$ and $\vec{n}_3 = 1/2\vec{v} \times \vec{u}$.
Two vectors along the last face are $\vec{u} - \vec{v}$ and $\vec{w} - \vec{v}$ and (twice) the last vector is

$$\begin{aligned} 2\vec{n}_4 &= \vec{u} - \vec{v}) \times (\vec{w} - \vec{v}) \\ &= (\vec{w} - \vec{v}) \times (\vec{v} - \vec{u}) \\ &= \vec{w} \times \vec{v} - \vec{w} \times \vec{u} + \vec{v} \times u \\ &= -\vec{v} \times \vec{w} - \vec{w} \times \vec{u} - \vec{u} \times v \\ &= -2\vec{n}_1 - 2\vec{n}_2 - 2\vec{n}_3. \end{aligned}$$

But then the sum

$$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0},$$

is indeed zero.
Here is a cute and compelling argument from physics, which seems worth mentioning. Imagine a beach ball made in the shape of the tetrahedron. If we blow the beach ball up, the air inside pushes on the walls of the beach ball. The force is directed outwards in the direction normal to each face. The pressure is equally distributed so the magnitude of the force is proportional to the area of each face. It follows that the forces on each side are the vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$ and $\vec{n}_4$. The beach ball certainly doesn't accelerate in any direction (let's ignore gravity) so the total force, that is, the sum of all the forces, must be zero.
4. (a) We compute the square of the matrix,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent.

(b) We have $\det AB = (\det A)(\det B)$. It follows that

$$\det A^n = (\det A)^n.$$

If $A^n = 0$ is the zero matrix, then the LHS is zero. But then

$$(\det A)^n = 0,$$

which can only happen if $\det A = 0$ to begin with.
(c) NO. Take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then A is nilpotent by (a) and it is easy to check that $B^2 = 0$, so B is also nilpotent. But the sum is

$$C = A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

There are a couple of ways to see that C is not nilpotent. For a start we could observe that

$$\det C = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

and use (b) to conclude that C cannot be nilpotent.
On the other hand,

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

so that

$$C^m = \begin{cases} C & \text{if } m \text{ is odd} \\ I_2 & \text{if } m \text{ is even.} \end{cases}$$

In particular no power of C is the zero matrix and so C is not nilpotent.
5. (a) $\vec{u} = A_\theta \hat{\imath} = \cos\theta \hat{\imath} + \sin\theta \hat{\jmath}$ and $\vec{v} = A_\theta \hat{\jmath} = -\sin\theta \hat{\imath} + \cos\theta \hat{\jmath}$.
(b) We simply multiply out

$$\begin{aligned} A_\theta A_\phi &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\phi\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = A_{\theta+\phi}. \end{aligned}$$

Geometrically all this says is that if you first rotate through an angle of ϕ and then rotate through and an angle θ, this is the same as rotating through a total angle of $\theta + \phi$.

(c) We first calculate the determinant

$$\det A_\theta = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$$

The inverse matrix is then

$$A_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Visibly this is the same as the transpose of A_θ.

$$A_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A_\theta^{-1}.$$

Geometrically all this says is that if want to undo the action of rotating through an angle of θ, simply rotate through an angle of θ in the opposite direction.

(d)

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(e) The determinants are -1, 1, 1 and -1. So the second and third matrices are rotation matrices and the first and fourth matrices are reflection matrices. A rotation matrix has the form

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

so matching the first columns of the second matrix we must have

$$\cos\theta = -\frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sin\theta = -\frac{1}{\sqrt{2}}.$$

Since both $\sin\theta$ and $\cos\theta$ are negative, we have an angle in the third quadrant. It follows that

$$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

So the second matrix represent rotation through

$$\frac{5\pi}{4}.$$

Matching the first column of the general rotation matrix and the third matrix, we get

$$\cos\theta = -\frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sin\theta = \frac{1}{\sqrt{2}}.$$

We must have an angle in the 2nd quadrant and

$$\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$

So the third matrix represents a rotation through $3\pi/4$.
The other matrices are a little bit more tricky, they represent a reflection. Pick an arbitrary reflection, let's say reflection in the y-axis,

$$(x, y) \longrightarrow (-x, y)$$

The corresponding reflection matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we multiply the first and fourth matrices by this matrix, we get rotation matrices

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Arguing as above, these two matrices represent rotations through $\pi/4$ and $7\pi/4$. So the first and fourth matrices represent rotation through $\pi/4$ followed by reflection in the y-axis and rotation through $7\pi/4$ followed by reflection in the y-axis.