SECOND MIDTERM
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name: ____________________

Signature: ____________________

Student ID #: ____________________

Recitation instructor: ____________________

Recitation Number+Time: ____________________

Problem	Points	Score
1	20	
2	15	
3	20	
4	15	
5	15	
6	15	
Total	100	

1. (20pts) Let $f(x, y) = 3xy^2 - x - y$.
(i) Find ∇f at $(2, 1)$.

Solution:
$\nabla f = \langle 3y^2 - 1, 6xy - 1 \rangle$. At $(2, 1)$ we have $(\nabla f)_{(2,1)} = \langle 2, 11 \rangle$.

(ii) Find the equation of the tangent plane to the graph of $f(x, y)$ at the point $(2, 1, 3)$.

Solution:
$z - 3 = 2(x - 2) + 11(y - 1)$. Rearranging, $2x + 11y - z = 12$.

(iii) Use linear approximation to estimate the value of $f(2.1, 0.9)$.

Solution: $\Delta f \approx f_x \Delta x + f_y \Delta y$. So, $\Delta f \approx 2 \cdot 0.1 + 11 \cdot -0.1 = -0.9$. $f(2.1, 0.9) \approx 2.1$.

(iv) Find the directional derivative of f at $(2, 1)$ in the direction of $\hat{\imath} + \hat{\jmath}$.

Solution: Direction is $\hat{u} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$. Directional derivative in direction $\hat{u}$ is

$$\left.\frac{df}{ds}\right|_{\hat{u}} = \frac{1}{\sqrt{2}} \langle 2, 11 \rangle \cdot \langle 1, 1 \rangle = \frac{13}{\sqrt{2}}.$$

2. (15pts) Let p be the point on the curve $x^2 - y^3 = 2.9$ which is closest to $(2, 1)$. Use the gradient to estimate the coordinates of p.

Solution: Let $f(x, y) = x^2 - y^3$. As $f(2, 1) = 3$, we want the point closest to $(2, 1)$ such that $\Delta f = -0.1$. So we want to move in the direction of greatest decrease in f.

$$\nabla f = \langle 2x, -3y^2 \rangle,$$

and so at $(x_0, y_0) = (2, 1)$ we have $\nabla f = \langle 4, -3 \rangle$. The direction of greatest decrease is

$$\hat{u} = \frac{1}{5}\langle -4, 3 \rangle.$$

If we go in this direction, the change is 5. So we need a displacement of

$$\frac{1}{250}\langle -4, 3 \rangle.$$

In other words we want

$$(1.984, 1.012).$$

3. (20pts) (i) Find the critical points of $w = \frac{1}{3}x^3 - x^2 + 2xy + y^2$, and determine the type of the critical point closest to the origin.

Solution: $f_x = x^2 - 2x + 2y$ and $f_y = 2x + 2y$. Setting these equal to zero we get $y = -x$ so that $x^2 = 4x$, that is, either $x = 0$ or $x = 4$. There are two critical points, $(0, 0)$ and $(4, -4)$. $(0, 0)$ is closest to the origin.

$$f_{xx} = 2x - 2, \qquad f_{xy} = 2 \qquad \text{and} \qquad f_{yy} = 2.$$

At $(0, 0)$, we have

$$A = f_{xx}(0,0) = -2 \qquad B = f_{xy}(0,0) = 2 \qquad \text{and} \qquad C = f_{yy}(0,0) = 2.$$

$AC - B^2 = -4 - 4 < 0$. We have a saddle point.

(ii) Find the point of the region $y \geq 0$, $y \leq x$ at which w is the smallest. Justify your answer.

Solution: The boundary of the region is the positive x-axis and the line $y = x$ in the first quadrant. When x or y goes to infinity w goes to infinity. The only critical points belonging to the region are on the boundary. So the minimum is on the boundary.

If we plug in $y = x$, we get

$$g(x) = f(x, x) = \frac{x^3}{3} + 2x^2.$$

$g_x = x^2 + 4x$. Either $x = 0$ or $x = -4$. At $x = 0$, $f(0, 0) = g(0) = 0$. $x = -4$ is not a point of the region.

If we plug in $y = 0$, we get

$$h(x) = f(x, 0) = \frac{x^3}{3} - x^2.$$

$h_x = x^2 - 2x$. This has two critical points, one at $x = 0$ and one at $x = 2$. At $x = 0$, $h(0) = f(0, 0) = 0$ and $h(2) = f(2, 0) = 8/3 - 4 = -4/3$.

So the minimum is at $(2, 0)$.

4. (15pts) (i) Write down the equations to find the point on the surface $(x+y)z^2 = 1$ in the first octant, closest to the origin, using the method of Lagrange multipliers.

Solution: As usual we minimise the square of the distance and not the distance. So we want to

$$\text{minimise} \quad x^2 + y^2 + z^2 \qquad \text{subject to} \qquad (x + y)z^2 = 1.$$

Therefore

$$\begin{aligned} 2x &= \lambda z^2 \\ 2y &= \lambda z^2 \\ 2z &= \lambda 2z(x + y) \\ (x + y)z^2 &= 1. \end{aligned}$$

(ii) Solve these equations to find the closest point.

Solution: First note that if x, y or z goes to infinity then the distance goes to infinity.

Comparing the first two equations, we have $x = y$. Since $z \neq 0$, the third equation yields $\lambda = 1/(2x)$. Plugging this into the first equation gives $z^2 = 4x^2$, so that $z = 2x$ (all the variables are positive). Using the fourth equation we get $8x^3 = 1$ and so $x = 1/2$. So $(1/2, 1/2, 1)$ is the only extreme point. Hence $(1/2, 1/2, 1)$ is the closest point to the origin.

5. (15pts) Let $w = f(u, v)$ where $u = xy$ and $v = x/y$. Using the chain rule, express

$$\frac{\partial w}{\partial x} \qquad \text{and} \qquad \frac{\partial w}{\partial y}$$

in terms of x, y, f_u and f_v.

Solution:

$$\mathrm{d}u = y\,\mathrm{d}x + x\,\mathrm{d}y \qquad \text{and} \qquad \mathrm{d}v = \frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y.$$

So,

$$\begin{aligned} \mathrm{d}w &= f_u\,\mathrm{d}u + f_v\,\mathrm{d}v \\ &= f_u(y\,\mathrm{d}x + x\,\mathrm{d}y) + f_v\left(\frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y\right) \\ &= \left(f_u y + \frac{f_v}{y}\right)\mathrm{d}x + \left(f_u x - \frac{x f_v}{y^2}\right)\mathrm{d}y. \end{aligned}$$

Therefore

$$\frac{\partial w}{\partial x} = f_u y + \frac{f_v}{y} \qquad \text{and} \qquad \frac{\partial w}{\partial y} = f_u x - \frac{x f_v}{y^2}.$$

6. (15pts) The two surfaces $x^2 + y^3 - z^4 = 1$ and $z^3 + zx + xy = 3$ intersect along a curve for which y is a function of x. Find

$$\frac{dy}{dx} \qquad \text{at} \qquad (x_0, y_0, z_0) = (1, 1, 1).$$

Solution: We use the method of differentials.

$$2x\,\mathrm{d}x + 3y^2\,\mathrm{d}y - 4z^3\,\mathrm{d}z = 0 \qquad \text{and} \qquad (y+z)\,\mathrm{d}x + x\,\mathrm{d}y + (3z^2+x)\,\mathrm{d}z = 0.$$

At the point $(x_0, y_0, z_0) = (1, 1, 1)$, we have

$$2\,\mathrm{d}x + 3\,\mathrm{d}y - 4\,\mathrm{d}z = 0 \qquad \text{and} \qquad 2\,\mathrm{d}x + \mathrm{d}y + 4\,\mathrm{d}z = 0.$$

Adding both equations together, we eliminate $\mathrm{d}z$,

$$4\,\mathrm{d}x + 4\,\mathrm{d}y = 0.$$

Solving, we get

$$\frac{dy}{dx} = -1.$$