

6. VELOCITY AND ACCELERATION

A particle moving in space sweeps out a curve. The position vector

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \langle x(t), y(t), z(t) \rangle,$$

is naturally a function of time t . For example, the cycloid

$$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle.$$

The **velocity vector** is simply the derivative of the position vector with respect to time

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle.$$

Notice that velocity is a vector; it has a magnitude and a direction. In the case of the cycloid,

$$\vec{v}(t) = \langle 1 - \cos t, \sin t \rangle.$$

At time $t = 0$, the rotation and the motion of the wheel cancel

$$\vec{v}(0) = \langle 0, 0 \rangle,$$

and at time $t = \pi$

$$\vec{v}(\pi) = \langle 2, 0 \rangle,$$

they combine.

The magnitude of the velocity is the **speed** $|\vec{v}|$. For the cycloid, the speed is

$$|\langle 1 - \cos t, \sin t \rangle| = ((1 - \cos t)^2 + \sin^2 t)^{1/2} = \sqrt{2}(1 - \cos t)^{1/2}.$$

At $t = 0$ the speed is zero and at $t = \pi$ the speed is 2.

The **acceleration vector** is simply the derivative of the velocity vector with respect to time,

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

For the cycloid the acceleration vector is

$$\vec{a} = \langle \sin t, \cos t \rangle.$$

Question 6.1. *What is the speed?*

It is

$$\left| \frac{d\vec{r}}{dt} \right|.$$

It is not

$$\frac{d|\vec{r}|}{dt}.$$

For example, imagine going around a circle. Then $|\vec{r}|$ is constant, so that the second expression is zero. But you can speed up and slow down even if you are going around a circle.

Let s be the distance travelled along the path. Then

$$\frac{ds}{dt} = |\vec{v}|,$$

is the speed. So we can recover the distance travelled by integrating the speed. However it is not always so easy to do this. For the cycloid, one full revolution is

$$s = \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt.$$

If one uses the trigonometric identity,

$$1 - \cos t = 2 \sin^2(t/2),$$

one see that the integrand is

$$2 \sin(t/2),$$

at least when $\sin(t/2) \geq 0$. So the integral is

$$4 \int_0^{\pi/2} 2 \sin(t/2) dt = 4 [-4 \cos(t/2)]_0^{\pi/2} = 4 - 2^{3/2}.$$

Most of the time it is not possible to calculate the integral. s is called the **arclength parameter**.

The **unit tangent vector** is the direction of the velocity,

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}.$$

In fact

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \frac{ds}{dt} = \hat{T} \frac{ds}{dt} = |\vec{v}| \hat{T}.$$

Based on some astronomical observations of Tycho Brahe, Johannes Kepler formulated three laws of planetary motion. The second law states that the area swept out by the vector from the sun to the planet sweeps out equal area in equal time. Newton derived all of Kepler's laws from calculus and his universal theory of gravitation.

It is relatively easy to derive Kepler's second law using vector calculus.

The area swept out in time Δt is approximately

$$\frac{1}{2} |\vec{r} \times \Delta \vec{r}| \approx \frac{1}{2} |\vec{r} \times \vec{v}| \Delta t.$$

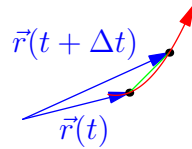


FIGURE 1. Area swept out in time Δt

So Kepler's second law may be restated as saying

$$\frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \vec{v}|,$$

is constant. Now $\vec{r} \times \vec{v}$ is perpendicular to the plane of motion, so the direction of the cross product $\vec{r} \times \vec{v}$ is constant. Putting all of this together, Kepler's second law says that the cross product

$$\vec{r} \times \vec{v}$$

is a constant vector.

The usual Leibniz rule applies to both the dot and the cross product.

$$\frac{d(\vec{a} \cdot \vec{b})}{dt} = \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b},$$

and

$$\frac{d(\vec{a} \times \vec{b})}{dt} = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}.$$

For dot products, we don't need to be careful about the order but the order is important for the cross product.

$$\begin{aligned} \vec{0} &= \frac{d(\vec{r} \times \vec{v})}{dt} \\ &= \vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}. \\ &= \vec{r} \times \vec{a} + \vec{v} \times \vec{v}. \\ &= \vec{r} \times \vec{a}. \end{aligned}$$

It follows that $r \times \vec{v}$ is constant if and only if the acceleration vector is parallel to \vec{r} . Note that Newton's second law has a vector form

$$\vec{F} = m\vec{a}.$$

So the acceleration \vec{a} is parallel to \vec{r} if and only if the force \vec{F} is parallel to \vec{r} . In other words, Kepler's second law is equivalent to the statement that the force is directed to the sun.