31. Stokes Theorem

Stokes' theorem is to Green's theorem, for the work done, as the divergence theorem is to Green's theorem, for the flux. Both are 3D generalisations of 2D theorems.

Theorem 31.1 (Stokes' Theorem)**.** *Let C be any closed curve and let S be any surface bounding C. Let $\vec{F}$ be a vector field on S.*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, \mathrm{d}S.$$

Note that S is an oriented surface. How do we orient S? We use the orientation on C.

If we drive along C, in the positive direction, with S on the left, then $\hat{n}$ should point upwards (with respect to the driver; that is to say, if you ask the driver to point to the roof of the car, this is the direction we should orient $\hat{n}$).

Put differently, we can use the right hand rule. If our index finger points along C, the middle finger points into S then the thumb points in the direction of $\hat{n}$.

Here are some examples:

(1) If C is the unit circle in the xy-plane, oriented counterclockwise and S is the upper hemisphere of the unit sphere, then $\hat{n}$ points outwards.
(2) If S is the half circular unit cylinder $x^2 + y^2 = 1$, $y \geq 0$, $0 \leq z \leq 1$, and C is the boundary curve, starting at $(1, 0, 0)$, going around to $(-1, 0, 0)$, up to $(-1, 0, 1)$, going around to $(1, 0, 1)$ and down to $(1, 0, 0)$, then $\hat{n}$ points outwards.
(3) If S is the cone with vertex at $(0, 0, 1)$ and base $x^2 + y^2 = 1$ in the xy-plane and we are orient the unit circle C counterclockwise, then $\hat{n}$ points outwards.

Suppose that C is a curve in the xy-plane, oriented counterclockwise. C bounds a region S in the xy-plane. If

$$\vec{F} = M\hat{\imath} + N\hat{\jmath},$$

then Green's theorem says

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_S \operatorname{curl} \vec{F}\,\mathrm{d}A = \int_S (N_x - M_y)\,\mathrm{d}A.$$

On the other hand, $\hat{n} = \hat{k}$, so that Stokes's theorem says

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_S (\nabla \times \vec{F}) \cdot \hat{n}\,\mathrm{d}A = \int_S (N_x - M_y)\,\mathrm{d}A.$$

So the two theorems are equivalent in this case.

Note one very interesting aspect of Stokes' theorem is that there are very many different surfaces S which bound the same curve C. For example, for the unit circle, the upper hemisphere bounds C, the lower hemisphere bounds C, the cone bounds C, and so on.

Proof of (31.1). We have already seen that if C and S lie in the xy-plane then Stokes' theorem reduces to Green's theorem.

If S and C live in an arbitrary plane, then imagine first translating the plane so that it contains the origin and then rotating it so that it is flat, that is, so it is the xy-plane.

The work done is invariant under both translation and rotation, so that the LHS of Stokes' theorem does not depend on the plane. The same is true for the RHS (this takes a little more justification). So (31.1) holds if S and C lie in any plane.

In the general case, we can subdivide S into small pieces (like a quilted blanket). As usual it is enough to prove (31.1) for each small quilt. If the piece is small enough, S and C live approximately in a plane and we can appeal to Green's theorem. □

Example 31.2. *Let's check that* (31.1) *holds in a special case. Let*

$$\vec{F} = z\hat{\imath} + x\hat{\jmath} + y\hat{k},$$

let C be the unit circle in the xy-plane and S be the paraboloid $z = 1 - x^2 - y^2$, $z > 0$.

We compute the LHS. We parametrise C by

$$x(t) = \cos t, \qquad y(t) = \sin t, \qquad z = 0 \qquad \text{where} \qquad 0 \leq t \leq 2\pi.$$

In this case

$$\mathrm{d}\vec{r} = \langle -\sin t, \cos t, 0\rangle \, \mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle 0, \cos t, \sin t\rangle.$$

So

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} \cos^2 t \, \mathrm{d}t = \left[\frac{1}{2}(1 + \frac{1}{2}\sin 2t)\right]_0^{2\pi} = \pi.$$

Now let's compute the RHS.

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{\imath} + \hat{\jmath} + \hat{k}.$$

The paraboloid

$$z = 1 - x^2 - y^2 = f(x, y)$$

is the graph of a function. So,

$$\mathrm{d}\vec{S} = \langle -f_x, -f_y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y = \langle 2x, 2y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y.$$

Note that this is the outwards orientation and that C is oriented counterclockwise, so C and S are compatibly oriented. Let R be the unit disk.

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,\mathrm{d}S = \iint_R (1 + 2x + 2y)\,\mathrm{d}A = \pi.$$

Here we used the fact that x is anti-symmetric about the y-axis, so that

$$\iint_R 2x\,\mathrm{d}A = 0.$$

Similarly

$$\iint_R 2y\,\mathrm{d}A = 0,$$

as y is anti-symmetric about the x-axis. Finally the area of R is π.