18.02 HOMEWORK #2, DUE THURSDAY SEPTEMBER 20TH

Part A (19 points)

(09/13) Read: Notes M.4 (pages 8-11), Book pages 798-800.
1H: 3abc, 7.
12.4: 23, 32, 50, 51.
1E: 1abcde, 2, 6.
(09/14) Read: pages 796-797, 10.4 to page 647.
1E: 3abc, 4,5.
1I: 1, 3abd, 5.
(09/18) Read: 12.5 to page 808; 12.6 to (7) on page 818; Notes K.
12.5: 2, 3, 4, 33.
1J: 1, 2, 3, 4, 5, 6, 9, 10.

Part B (23 points)

1. (Thursday, 7 points: 1+2+2+2) Cookies, doughnuts and croissants contain essentially the same ingredients (flour, sugar, eggs, butter) but in different amounts. For example, it takes 22 grammes of flour to make a cookie, 40g for a doughnut and 50g for a croissant. The amounts for each pastry can be encoded in a 4×3 matrix:

$$M = \begin{pmatrix} 22 & 40 & 50 \\ 18 & 10 & 3 \\ 5 & 4 & 5 \\ 10 & 10 & 22 \end{pmatrix},$$

where the rows are the amount of ingredient i ($i = 1$ for flour, $i = 2$ for sugar, $i = 3$ for eggs and $i = 4$ for butter) to make each pastry.
(a) Consider an assortment of x_1 cookies, x_2 doughnuts and x_3 croissants and use this to construct a column vector:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

What do the entries of $M\vec{x}$ correspond to?
(b) Each of the ingredients has a specific nutritional value: for example 100 grammes of flour contains 10g of protein, 76g of carbohydrates and 1g of fat. Proceeding similarly for the other ingredients, one can make a 3×4 matrix

$$N = \begin{pmatrix} 0.10 & 0 & .13 & 0 \\ .76 & 1 & .01 & 0 \\ .01 & 0 & .10 & .82 \end{pmatrix},$$

where the entries in the ith column represent the proportions of protein, carbohydrates and fat in ingredient i.

Give a matrix formula for the total nutritional value of the combination of pastries introduced in (a). (The answer should be in symbolic form: no need to evaluate numerically).

(c) Give a matrix formula which expresses the amounts x_i of each pastry one needs so as to abstorb y_1 g of protein, y_2 g of carbohydrates and y_3 g of fat. Express your answer in the form $\vec{x} = A\vec{y}$, and give both a symbolic formula for A and numerical values for its entries (use a calculator; Matlab, see the website for condensed instructions; your wrist watch; an abacus, etc).

(d) The recommended daily amounts of protein, carbohydrates and fat for a 2000 calorie diet are 50, 300 and 65 grammes, respectively. If you wanted to follow these guidelines while eating only cookies, doughnuts and croissants, how many of each pastries should one eat daily? What is wrong with your answer? Explain.

2. (Thursday, 4 points: 2+2) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a matrix of constants,

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

a column vector and λ a real number. If the matrix equation

$$A\vec{x} = \lambda\vec{x},$$

has a non-trivial solution (that is, a solution for which $\vec{x}$ is not the zero vector), then λ is called an eigenvalue of A.

(a) Show that A has either 2, 1 or no real eigenvalues, and

(b) give the respective inequalities connecting $a - d$ and bc that tell when each of the three cases in (a) holds.

3. (Thursday, 6 points: 3+3) Two lines in $\mathbb{R}^3$ are called skew lines, if they don't lie in the same plane and they don't intersect. Find the distance between two skew diagonals lying on adjacent sides of a unit cube, in two different ways:

(a) Find a pair of parallel planes, one of which contains one line, the other of which contains the other line and then find the distance between the two planes.

(b) (Friday) Find parametric equations for the two lines and find the times when a point P_1 on one line and a point P_2 on the other line are closest together. (Note that you need to parametrise the lines with two independent parameters).

(*Hint: for ease of calculation, put the cube in the first octant so that one vertex is at the origin. Choose the diagonals lying in the face* $x = 1$, *containing* $(1, 0, 0)$ *and lying in the face* $y = 1$, *containing* $(1, 1, 0)$. *For (b), what is the angle between the vector* $\overrightarrow{P_1P_2}$ *and either line?*)

4. (Friday, 6 points: 2+4) A mechanical device consists of two circular gears, one of radius 2 centred at $(0, -2)$ and the other of radius 1 centred at $(0, 1)$. The gear of radius 2 rotates clockwise at unit angular velocity (1 radian per second), while the gear of radius 1 rotates counterclockwise without slipping at the contact point. The two gears each carry a small

peg on their circumference, and these pegs are connected by an elastic band. Initially both pegs are at the origin.

(a) Write parametric equations for the motion of the midpoint P of the elastic band. (Use vectors; begin by expressing the position vector in terms of simpler vectors, then express each of them in terms of t).

(b) (Tuesday) Express the velocity of the point P as a function of t. Calculate the velocity and acceleration at $t = 0$.

PART C: 0 POINTS

Try to derive Kepler's first law:

"The orbit of every planet is an ellipse with the Sun at one of the two foci".