18.02 HOMEWORK #12, DUE THURSDAY DECEMBER 6TH

PART A (11 POINTS)

(11/29) Notes V11, V12; pages 1017-1018
6D/1$\underline{\text{a}}$bc$\underline{\text{d}}$, $\underline{2}$, $\underline{4}$, $\underline{5}$.
6E/1, 2, $\underline{\text{3ab(ii)}}$, $\underline{5}$.
(11/30) Notes V13; 15.7.
6F/1a$\underline{\text{b}}$, $\underline{2}$, 3, $\underline{5}$.
(12/04) Exam
(12/06) V14, V15.
6G/1
6H/1, 2.
(12/07) Review
(12/11) Review

PART B (20 POINTS)

1. (Thursday 6 points: 1+2+2+1) The Laplacian of a function f is the quantity

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$$

f is called harmonic if it satisfies Laplace's equation

$$\nabla^2 f = 0.$$

Let S be a closed surface bounding a region D. Suppose that f and g have continuous second partial derivatives inside D.
(i) Show that

$$\iint\limits_S\!\!\!\!\!\!\!\!\!\bigcirc\, \nabla f \cdot \mathrm{d}\vec{S} = \iiint\limits_D \nabla^2 f \, \mathrm{d}V.$$

(ii) Show that

$$\iint\limits_S\!\!\!\!\!\!\!\!\!\bigcirc\, f\nabla f \cdot \mathrm{d}\vec{S} = \iiint\limits_D (f\nabla^2 f + |\nabla f|^2) \, \mathrm{d}V.$$

(iii) Use (ii) to show that if f is harmonic in D and f is zero at every point of the boundary S of D, then f is zero everywhere in D.
(iv) Conclude that if f and g are harmonic functions in D and $f = g$ on the boundary S of D then $f = g$ everywhere in D.
2. (Thursday 5 points: 2+1+2)
(i) Compute, in terms of the constants a and b, the work done by the vector field

$$\vec{F} = (a\sin z + bxy^2)\hat{\imath} + 2x^2y\hat{\jmath} + (x\cos z - z^2)\hat{k}$$

along the portion of the helix

$$x = \cos t, \qquad y = \sin t \qquad \text{and} \qquad z = t,$$

from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

(ii) Compute $\operatorname{curl} \vec{F}$. For which values of a and b is $\vec{F}$ conservative?

(iii) Using the values of a and b from (ii), find a potential function f for $\vec{F}$ and check your answer to (i) using the fundamental theorem of calculus.

3. (Friday 9 points: 2+2+3+2) Consider a tetrahedron with vertices at $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$ and $P_3 = (1, 1, 0)$.

(i) Which orientation (specified by the order of the vertices) of the boundary curve of each face is compatible with the orientation of the tetrahedron given by the unit normals pointing outwards?

(ii) Compute the work done by the vector field

$$\vec{F} = yz\hat{\imath} - y^2\hat{k},$$

around the boundary of the face $P_0P_1P_3$ directly, using line integrals and the orientation from (i).

(iii) Use Stokes' theorem to compute the work done around each of the four faces, using the orientations from (i). (Note that you can avoid some calculation by re-using your answers from Hwk #11, Part B 4).

(iv) The sum of the four values should be zero. Explain why this is so in two different ways:

(a) geometrically, by considering the various line integrals that are being added together.

(b) by using the divergence theorem to compute the flux of the curl of $\vec{F}$ out of the tetrahedron.

Part C: 0 points