18.02 HOMEWORK #1, DUE THURSDAY SEPTEMBER 13TH

The homework is divided into parts A, B and C. Many of the problems in part A have solutions in the back of the book. These questions will be graded quite quickly (basically to check the answer is correct and that you didn't just copy the answer down). Try to solve these problems without looking at the answer first. The problems in part B don't have written solutions (yet) and will be graded more carefully. The problems in part C are purely for fun; no credit and no need to do them unless you want to.

Make sure you can do all the problems in parts A and B (and make sure you understand by checking your answers, either in the back of the book or after looking at the model answers after the hwk is due). Only the $\underline{\text{underlined}}$ problems should be handed in for credit. Dates indicate when the material you will need to answer the question will be covered in class.

Don't forget to quote your sources; see the syllabus for more details.

Part A (17 points)

Numbers like 12.1 refer to Edwards & Penny whilst numbers like 1A refer to the Supplementary Notes.

(09/06) Read: 12.1, 12.2.

12.1: 23, 31.

1A: 1, 5, $\underline{6}$, $\underline{7}$, 8, 9, $\underline{10}$.

1B: 1, $\underline{2}$, 11, $\underline{12}$, 13.

(09/07) Read: Notes D, 12.3.

12.2: $\underline{13}$ (for the vectors in 1 only), 39.

1B: 5a$\underline{\text{b}}$.

1C: 1, $\underline{2}$, 3, 4, $\underline{5\text{a}}$, 6, 7, 8.

1D: 1, $\underline{2}$, 3, 4, 5, $\underline{6}$, $\underline{7}$.

(09/11) Read: Notes M.1, M.2 (pages 1-7).

1F: 3, 5a$\underline{\text{b}}$, $\underline{8\text{a}}$, 9

1G: $\underline{3}$, $\underline{4}$, $\underline{5}$.

Part B (23 points)

1. (Thursday, 5 points: 2+1+1+1)

The eight vertices of a cube centered at $(0, 0, 0)$ of side length 2 are at $(\pm 1, \pm 1, \pm 1)$.

a) Find the four vertices of the cube, starting with $(1, 1, 1)$, that form a regular tetrahedron. Confirm your answer by finding the length of an edge and explaining why all edges have the same length.

b) A methane molecule consists of a hydrogen atom at each of the vertices of a regular tetrahedron and a carbon atom at the center. Find the 'bond angle', that is, the angle made by vectors from the carbon atom to two hydrogen atoms (use a calculator; round your answer).

c) Use the dot product to find the angle between two adjacent edges (sharing a common vertex) of the tetrahedron; and the angle between two opposite edges. Explain your answers using symmetry.

d) (Friday) Find the area of a face of the tetrahedron.

2. (Thursday, 3 points: 1+1+1) Consider a triangle in the plane with vertices P_1, P_2, and P_3.

a) Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ be the three vectors in the plane from the points P_1, P_2 and P_3 respectively to a point P. Express in terms of the dot product and these three vectors the condition that P is on the altitude of the triangle $P_1P_2P_3$ from the vertex P_1. (By altitude we mean the entire line through a vertex perpendicular to the opposite side, not just the segment from the vertex to the side.)

b) Assume that P is at the intersection of the altitudes from P_1 and P_2. Show that

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3.$$

c) Under the assumptions in (b), show that P is also on the altitude from P_3. (Hence all three altitudes meet in one point, called the orthocenter.)

3. (Friday, 3 points) For each face of a general tetrahedron, construct a vector perpendicular (aka orthogonal) to the face, which points outwards and has length equal to the area of the face. Show that the sum of these four vectors is the zero vector.

4. (Tuesday, 3 points: 1+1+1) A nilpotent matrix is a square matrix A such that $A^n = 0$ for some positive integer n.

(a) Show that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent.

(b) Show that if A is a nilpotent $m \times m$ matrix for some $m \geq 1$, then $\det A = 0$.

(c) If A and B are nilpotent matrices of the same size, must $A + B$ be nilpotent? (If YES, explain why. If NO, exhibit a pair of specific nilpotent matrices A and B such that $A + B$ is not nilpotent.)

5. (Tuesday, 9 points: 1+2+2+2+2) Orthogonal matrices are square matrices A that satisfy the identity $AA^T = I$ (I is the identity matrix). An equivalent definition of an orthogonal matrix property is that $A^TA = I$, because the left and right inverses of a square matrix are the same (see 1G-9b). The equation $AA^T = I$ says that the rows of A are perpendicular to each other and of unit length, whereas the equation $A^TA = I$ says that the columns of A are perpendicular to each other and of unit length. The geometric significance of orthogonal matrices is that multiplication by an orthogonal matrix preserves lengths of vectors and the absolute values of angles between them:

$$|A\vec{v}| = |\vec{v}| \qquad \text{and} \qquad |\angle(A\vec{v}, A\vec{w})| = |\angle(\vec{v}, \vec{w})|$$

There are two types of orthogonal matrices, rotations and reflections.

a) In $\mathbb{R}^2$, rotations are given by

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Find

$$\vec{u} = A_\theta \hat{\imath} \qquad \text{and} \qquad \vec{v} = A_\theta \hat{\jmath}$$

and draw a picture of $\vec{u}$ and $\vec{v}$ for $\theta = \frac{\pi}{4}$.

b) Use the addition formulas for sine and cosine to deduce that $A_\theta A_\phi = A_{\theta+\phi}$. Say in words what this matrix formula means about rotations.

c) Calculate A_θ^{-1}, and use this to verify that $A_\theta A_\theta^T = I$ (in other words, rotations are orthogonal matrices). Also verify that $A_\theta^{-1} = A_{-\theta}$, and give a geometric reason why this property holds.

d) Find the four orthogonal 2×2 matrices with first entry $a_{11} = -\frac{1}{\sqrt{2}}$. (*Hint*: try different signs. See 1F-9 and 1F-10.)

e) Next to each of the matrices in your list in part (d), draw what the matrix does to the letter F in the plane. Explain how the sign of the determinant of the matrix is related to the appearance of the transformed F.

Part C (0 points)

Pythagoras says that if we have a rectangle with sides a and b and diagonal c, then $c^2 = a^2 + b^2$. It is a natural question to look for rectangles where the three numbers (a, b, c) are all natural numbers; for example $(3, 4, 5)$, $(5, 12, 13)$.

So what happens for a box (aka a cuboid, aka a rectangular parallelepiped)? Suppose that that the three sides are a, b, c. There are three different faces diagonals and one big diagonal, making seven lengths.

Fix one length. Show that one can find a box where all but this length is a natural number. (In the end, writing a computer program which simply runs for ever until it finds a solution is probably the best way to solve this problem). It is an unsolved problem (aka due date ∞) whether one can find a box where all seven lengths are natural numbers.