6. Divisors

Definition 6.1. *We say that a scheme X is **regular in codimension one** if every local ring of dimension one is regular, that is, the quotient $\mathfrak{m}/\mathfrak{m}^2$ is one dimensional, where $\mathfrak{m}$ is the unique maximal ideal of the corresponding local ring.*

Regular in codimension one often translates to smooth in codimension one.

When talking about Weil divisors, we will only consider schemes which are

$(*)$ noetherian, integral, separated, and regular in codimension one.

Definition 6.2. *Let X be a scheme satisfying $(*)$. A **prime divisor** Y on X is a closed integral subscheme of codimension one.*

*A **Weil divisor** D on X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.*

Thus a Weil divisor is a formal linear combination $D = \sum_Y n_Y Y$ of prime divisors, where all but finitely many $n_Y = 0$. We say that D is **effective** if $n_Y \geq 0$.

Definition 6.3. *Let X be a scheme satisfying $(*)$, and let Y be a prime divisor, with generic point η. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K.*

*The **valuation ν_Y associated to** Y is the corresponding valuation.*

Note that as X is separated, Y is determined by its valuation. If $f \in K$ and $\nu_Y(f) > 0$ then we say that f has a **zero of order** $\nu_Y(f)$; if $\nu_Y(f) < 0$ then we say that f has a **pole of order** $-\nu_Y(f)$.

Definition-Lemma 6.4. *Let X be a scheme satisfying $(*)$, and let $f \in K^*$.*

$$(f) = \sum_Y \nu_Y(f) Y \in \operatorname{Div} X.$$

Proof. We have to show that $\nu_Y(f) = 0$ for all but finitely many Y. Let U be the open subset where f is regular. Then the only poles of f are along $Z = X - U$. As Z is a proper closed subset and X is noetherian, Z contains only finitely many prime divisors.

Similarly the zeroes of f only occur outside the open subset V where $g = f^{-1}$ is regular. $\square$

Any divisor D of the form (f) will be called **principal**.

Lemma 6.5. *Let X be a scheme satisfying $(*)$.*

The principal divisors are a subgroup of $\operatorname{Div} X$.

Proof. The map
$$K^* \longrightarrow \operatorname{Div} X,$$
is easily seen to be a group homomorphism. $\square$

Definition 6.6. *Two Weil divisors D and D' are called* ***linearly equivalent****, denoted $D \sim D'$, if and only if the difference is principal. The group of Weil divisors modulo linear equivalence is called the* ***divisor Class group****, denoted $\operatorname{Cl} X$.*

We will also denote the group of Weil divisors modulo linear equivalence as $A_{n-1}(X)$.

Proposition 6.7. *If k is a field then*
$$\operatorname{Cl}(\mathbb{P}^r_k) \simeq \mathbb{Z}.$$

Proof. Note that if Y is a prime divisor in $\mathbb{P}^r_k$ then Y is a hypersurface in $\mathbb{P}^n$, so that $I = \langle G \rangle$ and Y is defined by a single homogeneous polynomial G. The degree of G is called the degree of Y.

If $D = \sum n_Y Y$ is a Weil divisor then define the degree $\deg D$ of D to be the sum
$$\sum n_Y \deg Y,$$
where $\deg Y$ is the degree of Y.

Note that the degree of any rational function is zero. Thus there is a well-defined group homomorphism
$$\deg\colon \operatorname{Cl}(\mathbb{P}^r_k) \longrightarrow \mathbb{Z},$$
and it suffices to prove that this map is an isomorphism. Let H be defined by X_0. Then H is a hyperplane and H has degree one. The divisor $D = nH$ has degree n and so the degree map is surjective. One the other hand, if $D = \sum n_i Y_i$ is effective, and Y_i is defined by G_i,
$$(\prod_i G^{n_i}/X_0^d) = D - dH,$$
where d is the degree of D. $\square$

Example 6.8. *Let C be a smooth cubic curve in $\mathbb{P}^2_k$. Suppose that the line $Z = 0$ is a flex line to the cubic at the point $P_0 = [0 : 1 : 0]$. If the equation of the cubic is $F(X, Y, Z)$ this says that $F(X, Y, 0) = X^3$. Therefore the cubic has the form $X^3 + ZG(X, Y, Z)$. If we work on the open subset $U_3 \simeq \mathbb{A}^2_k$, then we get*
$$x^3 + g(x, y) = 0,$$
where $g(x, y)$ has degree at most two. If we expand $g(x, y)$ as a polynomial in y,
$$g_0(x)y^2 + g_1(x)y + g_2(x),$$

then $g_0(x)$ must be a non-zero scalar, since otherwise C is singular (a nodal or cuspidal cubic). We may assume that $g_0 = 1$. If we assume that the characteristic is not two, then we may complete the square to get

$$y^2 = x^3 + g(x),$$

for some quadratic polynomial $g(x)$. If we assume that the characteristic is not three, then we may complete the cube to get

$$y^2 = x^3 + ax + ab,$$

for some a and $b \in k$.

Now any two sets of three collinear points are linearly equivalent (since the equation of one line divided by another line is a rational function on the whole $\mathbb{P}^2_k$). In fact given any three points P, Q and P' we may find Q' such that $P + Q \sim P' + Q'$; indeed the line $l = \langle P, Q\rangle$ meets the cubic in one more point R. The line $l' = \langle R, P'\rangle$ then meets the cubic in yet another point Q'. We have

$$P + Q + R \sim P' + Q' + R'.$$

Cancelling we get

$$P + Q \sim P' + Q'.$$

It follows that if there are further linear equivalences then there are two points P and P' such that $P \sim P'$. This gives us a rational function f with a single zero P and a single pole P'; in turn this gives rise to a morphism $C \longrightarrow \mathbb{P}^1$ which is an isomorphism. It turns out that a smooth cubic is not isomorphic to $\mathbb{P}^1$, so that in fact the only relations are those generated by setting two sets of three collinear points to be linearly equivalent.

Put differently, the rational points of C form an abelian group, where three points sum to zero if and only if they are collinear, and P_0 is declared to be the identity. The divisors of degree zero modulo linear equivalence are equal to this group.

It is interesting to calculate the Class group of a toric variety X, which always satisfies $(*)$. By assumption there is a dense open subset $U \simeq \mathbb{G}_m^n$. The complement Z is a union of the invariant divisors.

Lemma 6.9. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where k is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \operatorname{Div}(X) \longrightarrow \operatorname{Div}(U).$$

If $Y' \subset U$ is a prime divisor, then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$, then the image of Y in $\operatorname{Div}(U)$ is equal to $(f|_U)$. If $Z = Z' \cup \bigcup_{i=1}^k Z_i$ where Z' has codimension at least two, then the map which sends $(m_1, m_2, \dots, m_k)$ to $\sum m_i Z_i$ generates the kernel. □

Example 6.10. *Let $X = \mathbb{P}^2_k$ and C be an irreducible curve of degree d. Then $\operatorname{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\operatorname{Cl}(\mathbb{A}^n_k) = 0$.*

It follows by (6.9) that there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_k$ it follows that $\operatorname{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel K. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda\chi^u$, where $\lambda \in k^*$ and $u \in M$.

It follows that there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

Lemma 6.11. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone σ, where σ spans $N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\operatorname{ord}_D(\chi^u) = \langle u, v\rangle$. In particular

$$(\chi^u) = \sum_i \langle u, v_i\rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. Using this, we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. □

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Example 6.12. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1, 0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0, 1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u, v, w]}{\langle v^2 - uw \rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1, 0)$ and $f_2 = (0, 1)$ then

$$(\chi^{f_1}) = D_1 - D_3 \qquad \text{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$.

We now turn to the notion of a Cartier divisor.

Definition 6.13. *Given a ring A, let S be the multiplicative set of non-zero divisors of A. The localisation A_S of A at S is called the* ***total quotient ring*** *of A.*

Given a scheme X, let $\mathcal{K}$ be the sheaf associated to the presheaf, which associates to every open subset $U \subset X$, the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. $\mathcal{K}$ is called the ***sheaf of total quotient rings****.*

Definition 6.14. *A* ***Cartier divisor*** *on a scheme X is any global section of $\mathcal{K}^*/\mathcal{O}_X^*$.*

In other words, a Cartier divisor is specified by an open cover U_i and a collection of rational functions f_i, such that f_i/f_j is a nowhere zero regular function.

A Cartier divisor is called **principal** if it is in the image of $\Gamma(X, \mathcal{K}^*)$. Two Cartier divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$, if and only if the difference is principal.

Definition 6.15. *Let X be a scheme satisfying $(*)$. Then every Cartier divisor determines a Weil divisor.*

Informally a Cartier divisor is simply a Weil divisor defined locally by one equation. If every Weil divisor is Cartier then we say that X

is **factorial**. This is equivalent to requiring that every local ring is a UFD; for example every smooth variety is factorial.

Definition-Lemma 6.16. *Let X be a scheme.*

The set of invertible sheaves forms an abelian group $\operatorname{Pic}(X)$, *where multiplication corresponds to tensor product and the inverse to the dual.*

Definition 6.17. *Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}$ by taking the subsheaf generated by f_i^{-1} over the open set U_i.*

Proposition 6.18. *Let X be a scheme.*

(1) *The association $D \longrightarrow \mathcal{O}_X(D)$ defines a correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$.*
(2) *$\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$, as abstract $\mathcal{O}_X$-modules.*
(3) *Two Cartier divisors D_1 and D_2 are linearly equivalent if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$, as abstract $\mathcal{O}_X$-modules.*

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. By (6.18) it suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)) \subset H^0(X, \mathcal{O}_X) = A_\sigma.$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

It follows that $I = A_\sigma \chi^u$, so that $D = (\chi^u)$ is principal. In particular, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 6.19. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. We have already seen* (6.12) *that the class group is $\mathbb{Z}_2$.*

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \,|\, \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (that is, given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = \sum a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$, where v_i is the primitive generator of the ray corresponding to D_i.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by u. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X),$$

is an embedding. It is an interesting to compare $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$ on a toric variety. Denote by $\mathrm{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 6.20. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_T(X) \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^s \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

In particular

$$\rho(X) = \mathrm{rank}(\mathrm{Pic}(X)) \leq \mathrm{rank}(\mathrm{Cl}(X)) = s - n.$$

Further $\mathrm{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If L is an invertible sheaf then $L|_U$ is trivial. Suppose that $L = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier and exactness of the top row is easy.

Finally, $\mathrm{Pic}(X)$ is subgroup of the direct sum of $M/M(\sigma)$ and each of these is a lattice, whence $\mathrm{Pic}(X)$ is torsion free. $\square$

Finally we end with an example to illustrate some of the difficulties of working with varieties which are not regular in codimension one.

Example 6.21. *Let $C \subset \mathbb{P}^2_k$ be the nodal cubic $ZY^2 = X^3 + X^2Z$, so that in the affine piece $U_3 \simeq \mathbb{A}^2_k$, $C \cap U_3$ is given by $y^2 = x^2 + x^3$. Let N be the node. Note that if D is a Weil divisor whose support does not contain N then D is automatically a Cartier divisor. As in the case of the smooth cubic, if P, Q, R and P', Q' and R' are two triples of collinear points on C (none of which are N), then $P + Q + R \sim P' + Q' + R'$.*

Now we already know that the nodal cubic is not isomorphic to $\mathbb{P}^1$. This implies that if P and P' are two smooth points of C then P and P' are not linearly equivalent. It follows, with a little bit of work, that all linear equivalences on C are generated by the linear equivalences above.

The normalisation of C is isomorphic to $\mathbb{P}^1$; on the affine piece where $Z \neq 0$ the normalisation morphism is given as $t \longrightarrow (t^2 - 1, t(t^2 - 1))$. The inverse image of the node N contains two points of $\mathbb{P}^1$ and it follows that $C - \{N\}$ is isomorphic to $\mathbb{G}_m$. In fact one can check that this is an isomorphism of algebraic groups, where the group law on $C - \{N\}$ is given by declaring three collinear points to sum to zero.

There is an exact sequence of groups,

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the first map sends P to $P - P_0 = [0 : 1 : 0]$, and the second map is the degree map which sends $D = \sum n_i P_i$ to $\sum n_i$.

Note that even though we can talk about Weil divisors on C, it only makes sense to talk about linear equivalences of Weil divisors supported away from N. Indeed, the problem is that any line through N cuts out $2N + R$, where R is another point of C. Varying the line varies R but fixes $2N$. In terms of Cartier divisors, a line through N (and not tangent to a branch) is equivalent to a length two scheme contained in the line. As we vary the line, both R and the length two scheme vary.

It is interesting to consider what happens at the level of invertible sheaves. Consider an invertible sheaf L on C which is of degree zero, that is, consider an invertible sheaf which corresponds to a Cartier *divisor D of degree zero. If $\pi\colon \mathbb{P}^1 \longrightarrow C$ is the normalisation map then*

$$\pi^* L = \pi^* \mathcal{O}_C(D) = \mathcal{O}_{\mathbb{P}^1}(\pi^* D),$$

has degree zero (to pullback a Cartier divisor, just pullback the defining equations. It is easy to check that this commutes with pullback of the sheaf). Since $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$, *$\pi^* L \simeq \mathcal{O}_{\mathbb{P}^1}$, the trivial sheaf. Now to get a sheaf on C we have to glue the two local rings over the inverse image N_1 and N_2 of N. The only isomorphisms of two such local rings are $\mathbb{G}_m$ acting by scalar multiplication (this is particularly transparent if one thinks of a invertible sheaf as a line bundle; in this case we are just identifying two copies of a one dimensional vector space) and this is precisely the kernel of the degree map on $\mathbb{P}^1$).*

There is a similar picture for the cuspidal cubic, given as $Y^2 Z = X^3$. The only twist is that $C - \{N\}$, where N is the cusp, is now a copy of $\mathbb{G}_a$.