## 12. Resolution of singularities II

The only reason why the approach sketched at the end of lecture 11 does not work in complete generality is that the induction breaks down. In dimension $n$, we use *embedded* resolution of singularities in dimension $n-1$. In other words, we only prove that every quasi-projective surface is birational to a smooth projective surface. But in dimension three, to get resolution of threefolds, we need to know that every divisor in a smooth threefold is birational to a divisor with global normal crossings.

So let's examine the problem of embedded resolution. As a warm up, let's look at the problem of embedded resolution of curves. We start with a smooth surface $S$ and a divisor $B = \sum B_i$ on $S$ (so that the prime components $B_1, B_2, \dots, B_k$ are curves) and we want to find a birational morphism

$$\pi \colon T \longrightarrow S,$$

such that the sum of the strict transform of the prime components of $B$ and of the exceptional divisors is a divisor with global normal crossings.

There is one way that the case of surfaces is significantly easier than in higher dimensions. At every step, we only need to choose which points to blow up on $S$. This means the problem is entirely local over every point of $S$ (this is very far from being the case in higher dimensions; more about this later).

Given that the problem is local, we may use the Weirstrass polynomial to keep track of the situation. Given any point $p \in S$, we pick local coordinates $x$ and $y$ so that $B$ is given by the zeroes of

$$y^{\mu} + a_{\mu-2}(x)y^{\mu-2} + a_{\mu-3}(x)y^{\mu-3} + \dots + a_1(x)y + a_0(x),$$

where $a_0(x), a_1(x), \dots, a_{\mu-2}(x)$ are analytic functions of the complex variable $x$. By assumption the multiplicity of $a_i(x)$ at 0 is at least $\mu - i$.

Now consider what happens when we blow up $S$ at the point $p$. Then

$$S_1 = \operatorname{Bl}_p S \subset S \times \mathbb{P}^1.$$

Suppose that we put coordinates $[S : T]$ on $\mathbb{P}^1$, so that $S_1$ is defined by $xT = yS$. Then the blow up is covered by two coordinate patches $S \neq 0$ and $T \neq 0$. If $S \neq 0$, then $y = xt$, and coordinates upstairs are given by $(x, t)$. The strict transform of $B$ is given by

$$t^{\mu} + \frac{a_{\mu-2}(x)}{x^2}y^{\mu-2} + \frac{a_{\mu-3}(x)}{x^3}t^{\mu-3} + \dots + a_1(x)t + \frac{a_0(x)}{x^{\mu}}.$$

If we put $y = t$ and

$$b_i(x) = \frac{a_i(x)}{x^{\mu-i}},$$

then the equation of the strict transform of $B$ is given by the zeroes of

$$y^\mu + b_{\mu-2}(x)y^{\mu-2} + b_{\mu-3}(x)y^{\mu-3} + \cdots + b_1(x)y + b_0(x),$$

where the multiplicity of $b_i(x)$ is at most the multiplicity of $a_i(x)$ minus $\mu - i$.

Suppose we consider what happens at a point $q$ lying over the point $p$, that is, a point $q$ of the exceptional divisor . We first check to see what happens over the unique point $[1 : 0]$ of the other coordinate patch. If $T \neq 0$, then $x = ys$ and it is easy to see that the strict transform of $B$ does not even pass through $[1 : 0]$. So we may assume that $S \neq 0$ and $y = xt$. Note that the multiplicity of the strict transform of $B$ at any point $q$ other than $(x, t) = (0, 0)$ is less than $\mu$. Indeed, let

$$\beta_i = b_i(0).$$

If we differentiate the equation

$$t^\mu + \beta_{\mu-2}y^{\mu-2} + \beta_{\mu-3}y^{\mu-3} + \cdots + \beta_1 t + \beta_0.$$

for the strict transform $\mu - 1$ times with respect to $t$, we get that

$$(\mu)!t = 0,$$

so that if $t = \alpha$ is a root of multiplicity $\mu$, then

$$t = 0.$$

Notice that this is heavily dependent on the fact the characteristic is zero. On the other hand, it is pretty clear that at the point $(x, t) = (0, 0)$, the situation is better, since the multiplicity of each one of the functions $b_i(x)$ dropped.

By induction on the multiplicity $\mu$ and the multiplicity of each $a_i(x)$, after finitely many blow ups, $f_1 \colon S_1 \longrightarrow S$, we reduce to the case when the strict transform $B_1$ of $B$ is smooth. Let $E_1$ be the sum of the exceptional divisors of $f_1$. Note that $(S_1, E_1)$ has global normal crossings. It remains to reduce to the case when $(S_1, E_1 + B_1)$ has normal crossings. Pick a point $p_1 \in S_1$, where $(S_1, B_1 = E_1)$ does not have normal crossings. Then $p_1$ is contained in exactly one component of $B_1$, since $B_1$ is smooth, and at most two components of $E_1$, since $E_1$ has global normal crossings. $B_1$ is tangent to at most component of $E_1$ (since if there are two components of $E_1$, then they have different tangents). Blowing up finitely many times, we reduce to the case when no component of $B_1$ is tangent to a component of $E_1$. At this point, the only problem is if three components contain the same point. Blowing up each of these points, we are done.

Now let us consider the situation in higher dimensions. From the case of embedded resolution of curves, it is clear that it is a good idea to keep track of some invariants. So what invariants should we consider?

**Example 12.1.** *Consider the surface*

$$X = (y^2 = zx^2 + x^3) \subset \mathbb{C}^3.$$

*This surface is called the* ***Whitney umbrella****. If we project,*

$$\pi \colon \mathbb{C}^3 \longrightarrow \mathbb{C},$$

*by sending $(x, y, z)$ to $z$, then we get a family of nodal curves*

$$y^2 = ax^2 + x^3,$$

*$a \neq 0$, degenerating to a cuspidal curve,*

$$y^2 = x^3.$$

*The singular locus of $X$ is the $z$-axis. Clearly the most singular point is the origin. So let's blow up the origin. Suppose that coordinates on the exceptional divisor $\mathbb{P}^2$ are $[A : B : C]$. The most important coordinate patch is $C \neq 0$, so that $x = az$ and $y = bz$.*

$$(bz)^2 - z(az)^2 - (az)^3 = z^2(b^2 - a^2z - a^3z).$$

*Replacing $a$ by $x$ and $b$ by $y$, we get*

$$y^2 - x^2z - x^3z,$$

*which hardly seems like progress.*

*In fact, we should blow up the $z$-axis. The blow up of $\mathbb{C}^3$ along the $z$-axis sits inside $\mathbb{C} \times \mathbb{P}^1$. Let's suppose that $\mathbb{P}^1$ has coordinates $[S : T]$. Then $xT = yS$ and there are two coordinate patches. The most relevant is given by $S \neq 0$, so that $y = xt$, and we get*

$$(xt)^2 - zx^2 - x^3 = x^2(t^2 - zx - x),$$

*and so the equation of the strict transform is*

$$t^2 - zx - x,$$

*which is smooth.*

*There is another way to look at all of this. If we forget the embedding of $X$ into $\mathbb{C}^3$, and consider the normalisation $\nu \colon X^\nu \longrightarrow X$ of $X$, then $X^\nu$ is smooth and fibres over $\mathbb{C}$ as well. The inverse image of the singular locus is a smooth curve $C$ which double covers $\mathbb{C}$. This morphism ramifies over the origin. If $Y \longrightarrow X$ denotes the blow up of $X$ at the origin, then the normalisation $Y^\nu \longrightarrow Y$ is simply given by* *blowing up $X^\nu$. It is then clear that no amount of blowing up points on $X$ will ever improve the situation.*

The moral of this example is that we don't need to be so careful to distinguish the most singular points. In fact the only real invariant we need to keep track of is the multiplicity.

Unfortunately it is also clear that we need to be quite careful how to choose the locus to blow up. For example consider

$$z^2 - x^3y^3.$$

The singular locus consists of the $x$ and $y$-axis. If we blow up either axis it is clear that we are making progress (generically along the $y$-axis we have $z^2 - x^3$, which is resolved in three steps by blowing up the origin). But we are not allowed to blow up an axis. The problem is that this is only the local analytic picture. Globally the singular locus might be a nodal cubic (for example). In this case it is not possible to blow up one axis, since globally blowing up one axis forces us to blow up the other axis. On the other hand, we cannot blow up both axes, since this locus is not smooth.

The only possible relevant locus we could blow up which is in the singular locus is the origin. On the blow up we have coordinates $(x, y, z) \times [A : B : C]$, and equations expressing the equality $[x : y : z] = [A : B : C]$. On the coordinate patch $A \neq 0$ we have $y = bx$, $z = cx$ so that

$$z^2 - x^3y^3 = c^2x^2 - b^3x^6 = x^2(c^2 - b^3x^4).$$

Changing variables we have $z^2 - x^3y^4$ which is surely worse than before. So how has the situation improved? The key thing is that the singular locus is given by $c = b = 0$ and $c = x = 0$. The locus $c = x = 0$ lies in the exceptional divisor; we created it ourselves, and so we know that this locus is algebraically irreducible and not just locally analytically irreducible. So we are allowed to blow up $c = x = 0$. In fact we are allowed to blow up $c = b = 0$, since the first blow up separated the $x$ and $y$-axis. Notice though that we must blow up the strict transform of the other axis (again, because globally it might be part of a single irreducible algebraic curve).

It is clear from this example, that we must keep track of the sequence of blow ups. Fortunately, it turns out we don't need to keep track of much of the history. All we really need to distinguish are components of the original variety and the exceptional locus (we also need to order the components by when they appear).

There is one simple way to make sure that we never get into trouble passing from the local picture to the global picture. Note that on the original surface, there is an obvious symmetry between $x$ and $y$. If every blow up respects this symmetry, we can never go wrong. We cannot chose to blow up the $x$-axis, since this is not symmetric. If we blow up the $x$-axis, then are allowed to blow up the strict transform of the $x$-axis, provided we also blow up the strict transform of the $y$-axis.