11. Resolution of singularities I

We start to consider the problem of resolution of singularities. At it most basic we are given a finitely generated field extension K/k and we would like to find a smooth projective variety X over k with function field K.

Before we get into a proof of resolution of singularities via smooth blow ups, we first describe some other ways to resolve singularities. Even though these methods don't always work, they introduce ideas and techniques which are of considerable independent interest.

Definition 11.1. *Let X be an integral scheme. We say that X is* ***normal*** *if all of the local rings $\mathcal{O}_{X,p}$ are integrally closed.*

The ***normalisation of*** *X is a morphism $Y \longrightarrow X$ from a normal scheme, which is universal amongst all such morphisms. If $Z \longrightarrow X$ is a morphism from a normal scheme Z, then there is a unique morphism $Z \longrightarrow Y$ which make the diagram commute:*

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X. \end{tikzcd}$$

One can always construct the normalisation of a scheme as follows. By the universal property, it suffices to construct the normalisation locally. If $X = \operatorname{Spec} A$, then $Y = \operatorname{Spec} B$, where B is the integral closure of A inside the field of fractions. Note that if X is quasi-projective variety then the normalisation $Y \longrightarrow X$ is a finite and birational morphism.

Definition 11.2. *Let X be a scheme. We say that X satisfies* ***condition*** *S_2 if every regular function defined on an open subset U whose complement has codimension at least two, extends to the whole of X.*

Lemma 11.3 (Serre's criterion)**.** *Let X be an integral scheme.*

Then X is normal if and only if it is regular in codimension one (condition R_1) and satisfies condition S_2.

Note that this gives a simple method to resolve singularities of curves. If C is a curve, the normalisation $C' \longrightarrow C$ is smooth in codimension one, which is to say that C' is smooth.

Note that lots of surface singularities are normal. For example, every hypersurface singularity is S_2, so that a hypersurface singularity is normal if and only if it is smooth in codimension one. Similarly, every quotient singularity is normal.

Before we pass on to other methods, it is interesting to write down some example of varieties which are R_1 but not normal, that is, which are not S_2.

Example 11.4. *Let S be the union of two smooth surfaces S_1 and S_2 joined at a single point p. For example, two general planes in $\mathbb{A}^4$ which both contain the same point p. Let $U = S - \{p\}$. Then U is the disjoint union of $U_1 = S_1 - \{p\}$ and $S_2 - \{p\}$, so U is smooth and the codimension of the complement is two. Let $f : U \longrightarrow k$ be the function which takes the value 1 on U_1 and the value 0 on U_2. Then f is regular, but it does not even extend to a continuous function, let alone a regular function, on S.*

Let C be a projection of a rational normal quartic down to $\mathbb{P}^3$, for example the image of

$$[S : T] \longrightarrow [S^4 : S^3T : ST^3 : T^4] = [A : B : C : D].$$

Let S be the cone over C. Then S is regular in codimension one, but it is not S_2. Indeed, $A/B = C/D$ is a regular function whose only pole is along $B = 0$ and $D = 0$, that is, only at $(0, 0, 0, 0)$ of S.

Beyond the dimension of the Zariski tangent space, perhaps the most basic invariant of any singular point is:

Definition 11.5. *Let $X \subset M$ be a subvariety of a smooth variety. The **multiplicity of** X **at** $p \in M$ is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^{\mu}$ where $\mathfrak{m}$ is the maximal ideal of M at p in $\mathcal{O}_{M,p}$ and $\mathcal{I}$ is the ideal sheaf of X in M.*

Note that this generalises the multiplicity of a hypersurface singularity. The multiplicity has two basic properties. X is smooth at p if and only if the multiplicity is one and the multiplicity is upper semicontinuous in families.

We next describe the method of Albanese. Start with $X \subset \mathbb{P}^n$. Now re-embed X by the very ample line bundle $\mathcal{O}_X(m)$, where m is very large, so that $X = X_0 \subset \mathbb{P}^r$, where r is large. Pick a point $p = p_0 \in X_0$, where the multiplicity is largest, to get $X_1 \subset \mathbb{P}^{r-1}$. Now pick a point $p_1 \in X_1$ of largest multiplicity and project down to get $X_2 \subset \mathbb{P}^{r-2}$. Continuing in this way, always projecting from a point of maximal mulitplicity, we construct $X_i \subset \mathbb{P}^{r-i}$.

Theorem 11.6. *If*

$$\deg X_0 < (n! + 1)(r + 1 - n),$$

then the Albanese algorithm stops with a variety X_k and a generically finite map $f_k \colon X_0 \dashrightarrow X_k$, such that either

(1) $\deg f_k \operatorname{mult}_p(X_k) \leq n!$, *or*
(2) X_k *is a cone and* $\deg f_k \leq n!$.

Corollary 11.7. *Assume that every variety of dimension at most* $n-1$ *is birational to a smooth projective variety.*

Then every projective variety is birational to a projective variety with singularities of multiplicity at most $n!$.

Note that this resolves singularities for curves, since $1! = 1$ and a point of multiplicity one is a smooth point of X. Even for surfaces we get down to points of multiplicity two, which are not so bad. Starting with threefolds, the situation is not nearly so rosy, especially when one realises that if f is a hypersurface singularity of arbitrary multiplicity, then the suspension of f, $x^2 + f$, is a hypersurface singularity of multiplicity two. It is pretty clear that resolving $x^2 + f$ entails resolving f.

Unfortunately it seems impossible to improve the bound given in (11.6).

We will need:

Theorem 11.8. *Let* $X \subset \mathbb{P}^r$ *be an irreducible projective variety of degree* d *and dimension* n.

If X *is not contained in a hyperplane, then*

$$d \geq r + 1 - n.$$

Proof of (11.6). By induction on k. Suppose that

$$\deg f_k \cdot \operatorname{mult}_p(X_k) \leq (n! + 1)(r - k + 1 - n).$$

Suppose that p is a point of maximal multiplicity μ. If X_k is a cone with vertex p, then there is nothing to prove. Otherwise let X_{k+1} be the closure of the image of p under projection, and let $\pi\colon X_k \dashrightarrow X_{k+1}$ be the resulting rational map. As X_k is not a cone over p, π is generically finite. We have

$$\deg \pi \cdot d_{k+1} = d_k - \mu.$$

If

$$\deg f_k \cdot \mu > n!,$$

then

$$\begin{aligned}
\deg f_{k+1} \cdot d_{k+1} &= \deg f_k \deg \pi \cdot d_{k+1} \\
&= \deg f_k \cdot d_k - \deg f_k \mu \\
&\leq \deg f_k \cdot d_k - (n! + 1) \\
&\leq (n! + 1)(r - k + 1 - n) - (n! + 1) \\
&\leq (n! + 1)(r - (k + 1) + 1 - n).
\end{aligned}$$

It follows that eventually either X_k becomes a cone or we get

$$\deg f_k \cdot \operatorname{mult}_p X_k \leq n!.$$

As $X_k \subset \mathbb{P}^{r-k}$ is not contained in a hyperplane, we have

$$d_k \geq (r - k + 1 - n).$$

It follows that if X_k is a cone, then

$$\deg f_k \leq n!. \qquad \square$$

Notice how truly bizarre this argument is; presumably projecting from a point will introduce all sorts of bad singularities (corresponding to secant lines and so on), but just by projecting from the point of maximal multiplicity works.

Example 11.9. *Let*

$$m_1 \leq m_2 \leq \cdots \leq m_r,$$

be a sequence of positive integers. Let C be the image of

$$t \longrightarrow (t^{m_1}, t^{m_2}, t^{m_3}, \dots, t^{m_r}),$$

inside $\mathbb{A}^r$. If we project from $(1, 0, 0, \dots, 0)$, then we get the image of

$$t \longrightarrow (t^{m_2-m_1}, t_2^{m_3-m_1}, t_3^{m_4-m_1}, \dots, t_r^{m_r-m_1}),$$

inside $\mathbb{A}^{r-1}$. It is intuitively clear that the projection of C is less singular than C, but it is hard to say exactly why; for example the multiplicity might go up.

Let us turn to the proof of (11.7). We will need:

Theorem 11.10 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^n$ has degree d.*

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + \dots,$$

is a polynomial of degree n, for m large enough, with the given leading term.

Proof. Let Y be a general hyperplane section. Then Y is a normal projective variety of degree d; indeed, Y is certainly regular in codimension one and one can check that Y is S_2. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$

By an easy induction, it follows that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing. $\square$

Definition 11.11. *Let X be a quasi-projective variety and let K be the function field of X. Let L/K be a finite field extension.*

*The **normalisation of X in L** is a finite morphism $Y \longrightarrow X$, where Y is a normal quasi-projective variety and the function field of Y is L.*

One can consruct Y in much the same way that one constructs the normalisation. It suffices to construct Y locally, in which case we may assume that $X = \operatorname{Spec} A$ is affine. In this case one simply takes $Y = \operatorname{Spec} B$, where B is the integral closure of A inside L.

Lemma 11.12. *Let $\pi\colon Y \longrightarrow X$ be a finite morphism.*

If $\pi(q) = p$, then

$$\operatorname{mult}_q Y = \deg \pi \cdot \operatorname{mult}_p X.$$

Proof of (11.7). By (11.10) we may pick m sufficiently large such that if

$$\deg X_0 \subset \mathbb{P}^r$$

is the embedding given $\mathcal{O}_X(m)$, then

$$d_0 \leq (n! + 1)(r + 1 - n).$$

By (11.6) we may find a generically finite morphism $f\colon X \dashrightarrow W$ such that either

$$\deg f \operatorname{mult}_w W \leq n!,$$

or W is a cone and

$$\deg f \leq n!.$$

If W is a cone, then W is birational to a product $\mathbb{P}^1 \times W'$. By our induction hypothesis, W' is birational to a smooth projective variety W''. Then W is birational to $W'' \times \mathbb{P}^1$. Replacing W by $W'' \times \mathbb{P}^1$, we may assume that W is smooth.

Let $\pi\colon Y \longrightarrow W$ be the normalisation of W in the field $L = K(X)/K(W)$. Then Y is birational to X and $\deg f = \deg \pi$. By (11.12),

$$\operatorname{mult}_y Y \leq n!.$$

$\square$

Another intriguing method was proposed by Nash:

Definition 11.13. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. The **Gauss map** is the rational map*

$$X \dashrightarrow \mathbb{G}(n, N) \qquad \text{given by} \qquad x \longrightarrow T_x X,$$

which sends a point to its (projective) tangent space.

*The **Nash blow up** is given by taking the graph of this rational map.*

Conjecture 11.14. *We can always resolve any variety by successively taking the Nash blow up and normalising.*

Despite the very appealing nature of this conjecture (consider for example the case of curves, when we don't even need to normalise) we only know (11.14) in very special cases. The one very nice feature of the Nash blow up is that it does not involve any choices. Unfortunately it is known that one needs to normalise, and this messes up any sort of induction.

If X is a toric variety there is a pretty simple method to resolve singularities. First subdivide the cone until X is simplicial. It is not too hard to argue that one can resolve any simplicial toric variety (one only needs to keep track of a simple invariant). One subtle point is to make sure that as one improves one cone, then another cone does not become worse.

There is a pretty straightforward way to resolve the singularities of a quasi-projective surface S. It does not hurt to assume that S is projective. Replacing S by its normalisation, we may assume that S is normal. First embed S into $\mathbb{P}^n$. By repeatedly projecting from a point, we may express S as a large degree cover of $\mathbb{P}^2$, $\pi\colon S \longrightarrow \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be the branch locus of π, the locus where two or more points come together.

Take an embedded resolution of $(\mathbb{P}^2, B)$. This is a birational morphism $f\colon N \longrightarrow \mathbb{P}^2$ such that the total transform C of B is a divisor with global normal crossings. Let T be the normalisation of the fibre product $N \underset{\mathbb{P}^2}{\times} S$. We have a commutative diagram,

$$\begin{array}{ccc} T & \xrightarrow{g} & S \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\pi} \\ N & \xrightarrow{f} & \mathbb{P}^2. \end{array}$$

Now $\psi\colon T \longrightarrow N$ only ramifies over C, which is a divisor with normal crossings. Consider the field extension $M = \mathbb{C}(T)/\mathbb{C}(N) = K$. This is not necessarily Galois; let L/M be the Galois closure, so that L/M and L/K are Galois extensions.

Let R be the normalisation of T inside L. If G is the Galois group of L/K, then G acts on R and $N = R/G$. Similarly if $H \subset G$ is the Galois group of L/M, then $T = R/H$. As a warm up, suppose first that R is smooth. Then T has quotient singularities, and we have already seen that it is easy to resolve the singularities of T. In fact, T has cyclic quotient singularities.

Definition 11.15. *Let $f\colon X \longrightarrow Y$ be a finite morphism. We say that f is* ***Galois****, if there is a finite group G acting on X such that f is the quotient map.*

Consider $R \longrightarrow N$. This morphism is Galois. Locally we have a Galois cover of $\mathbb{C}^2$, only ramified over the x and y-axis. Topologically we have an unramified Galois cover of the complement, a torus. Such covers are classified by the fundamental group,

$$\pi_1(\mathbb{C}^{*2}, 1) = \mathbb{Z} \oplus \mathbb{Z}.$$

A finite cover is given by a cylic quotient,

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_a \oplus \mathbb{Z}_b.$$

Let $n = ab$. Then the map

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad \text{given by} \qquad (x, y) \longrightarrow (x^n, y^n),$$

is a cover given by the following quotient of the fundamental group

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_n.$$

As

$$\mathbb{Z}_n \oplus \mathbb{Z}_n \longrightarrow \mathbb{Z}_a \oplus \mathbb{Z}_b,$$

factors the first map, it follows that any Galois cover

$$X \longrightarrow \mathbb{C}^2,$$

is itself a quotient of

$$\mathbb{C}^2 \longrightarrow X,$$

which only ramifies along the x and y-axis. So R has cyclic quotient singularities. It is easy to resolve R, preserving the action of G. The map $R \longrightarrow R/H$ is Galois and there is a birational morphism $R/H \longrightarrow T$. Finally, if we resolve the cyclic quotient singularities R/H, then we are done.