THIRD PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: **MODEL ANSWERS**

Signature: ______________________

Recitation Time: ____________

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

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1. (20 pts) For what values of \( \lambda, \mu \) and \( \nu \) does the function \( f : \mathbb{R}^3 \rightarrow \mathbb{R}, \)

\[
f(x, y, z) = \lambda x^2 + \mu xy + y^2 + \nu z^2,
\]

have a non-degenerate local minimum at \((0, 0, 0)\)?

\[
\begin{align*}
Df &= (2\lambda x + \mu y, 2y + \mu x, 2\nu z) \\
Hf &= \begin{pmatrix}
2\lambda & \mu & 0 \\
\mu & 2\nu & 0 \\
0 & 0 & 2\nu
\end{pmatrix}
\end{align*}
\]

\[
d_1 = 2\lambda, \quad d_2 = 4\lambda - \mu^2, \quad d_3 = 2\nu \cdot d_2.
\]

Minimum: \( d_1 > 0, \quad d_2 > 0, \quad d_3 > 0. \)

So \( \lambda > 0, \quad \mu^2 < 4\lambda, \quad \nu \neq 0. \)
2. (20pts) Let $f: \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = 2x + y - z$
(i) Show that $f$ has a global minimum on the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$.

$$K = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 3z^2 = 6 \}$$

is closed and bounded. So $K$ is compact.

If $f$ is continuous, $K$ is compact $\implies f$ has a global minimum.

(ii) Find this minimum.

Consider $h: \mathbb{R}^4 \to \mathbb{R}$ given by

$$h(x, y, z, \lambda) = 2x + y - z \Rightarrow \lambda(x^2 + 2y^2 + 3z^2 - 6)$$

Critical pts of $h$:

$$x = \frac{1}{x} \quad x = 4y = -6z$$

$$4y = \frac{1}{x} \quad y = -\frac{3z}{2}, \quad x = -6z.$$ 

$$-6z = \frac{1}{x}.$$

$$x^2 + 2y^2 + 3z^2 = 6 \quad z^2 \left(3 + \frac{9}{2} + 36\right) = 6$$

$$z^2 \left(2 + \frac{24}{2} \right) = 4 \quad z = \pm \sqrt{2} \sqrt{29} \quad f(p) = \left(2 + 3 + 2\right) \frac{1}{29}$$

$\min$ value: square root

$$P = \left(\frac{-12}{29}, \frac{-3}{29}, \frac{2}{29}\right) = -\sqrt{29}.$$
3. (20pts)
   (i) Draw a picture of the region of integration of

   \[ \int_0^1 \int_{1+x}^{\sqrt{9-x^2}} dy \, dx. \]

   

   (ii) Change the order of integration of the integral.

   \[ \int_1^2 \int_{y^{-1}}^{\sqrt{9}} dx \, dy + \int_1^2 \int_0^{\sqrt{9-y^2}} dx \, dy + \int_0^3 \int_{\sqrt{9-y^2}}^{\sqrt{9}} dx \, dy. \]
4. (20pts) Let \( W \) be the region inside the two cylinders \( x^2 + y^2 = 1 \) and \( y^2 + z^2 = 1 \). Set up an integral to calculate the volume of \( W \) and calculate this integral.

View as a region of type 2.

\[
\text{vol}(W) = \iiint_W dx \, dy \, dz
\]

\[
= \int_{-1}^{1} \left( \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left( \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \right) dz \right) dy
\]

\[
= 2 \int_{-1}^{1} \left( \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-y^2} \, dz \right) dy
\]

\[
= 4 \int_{-1}^{1} 1 - y^2 \, dy
\]

\[
= 4 \left[ y - \frac{y^3}{3} \right]_{-1}^{1}
\]

\[
= 8 \left( 1 - \frac{1}{3} \right) = \frac{16}{3}.
\]
5. (20 pts) Let $D$ be the region in the first quadrant bounded by the curves $y^2 = x$, $y^2 = 2x$, $xy = 1$ and $xy = 4$.

(i) Find $du \, dv$ in terms of $dx \, dy$, where $u = \frac{y^2}{x}$ and $v = xy$.

\[
\frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{y}{x} & -x \end{vmatrix} = -\frac{y^2}{x^2} \cdot \frac{x}{x} - \frac{2y}{x} \cdot \frac{x}{x} = -\frac{3y^2}{x^2}
\]

\[
\frac{\partial (x,y)}{\partial (u,v)} = -\frac{x}{3y^2} \quad dx \, dy = (3u)^{-1} \, du \, dv.
\]

(ii) Set up an integral to calculate the area of the region $D$ and calculate this integral.

\[
\iint_D \, dx \, dy = \int_{1}^{2} \left( \int_{1}^{u} \frac{du}{3u} \right) \, dv
\]

\[
= \frac{1}{3} \int_{1}^{4} \left[ \ln u \right]_1^2 \, dv
\]

\[
= \frac{\ln 2}{3} \int_{1}^{4} \, dv
\]

\[
= \ln 2.
\]