FIRST PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: **MODEL ANSWERS**

Signature: _______________________

Recitation Time: ___________________

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. (20pts) (i) Let \( \vec{u} \) and \( \vec{v} \) be two vectors. Show that the vectors 
\[ \vec{a} = \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u} \] 
and \( \vec{b} = \|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u} \) are orthogonal.

We check that \( \vec{a} \cdot \vec{b} = 0 \).

\[
\vec{a} \cdot \vec{b} = \left( \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u} \right) \left( \|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u} \right)
= \|\vec{u}\|^4 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2
= 0.
\]

So \( \vec{a} \) and \( \vec{b} \) are orthogonal.

(ii) Show that the vector \( \vec{a} = \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u} \) bisects the angle between \( \vec{u} \) and \( \vec{v} \).

We want to check \( \theta = \phi \).

It suffices to check \( \alpha = \beta \).

As \( \theta = 90 - \alpha \) and \( \phi = 90 - \beta \).

Now \( \vec{v} \cdot \vec{b} = \|\vec{v}\| \|\vec{b}\| \cos \alpha \)
\( \vec{u} \cdot \vec{b} = \|\vec{u}\| \|\vec{b}\| \cos \beta \).

So it suffices to check \( \|\vec{v}\| \vec{v} \cdot \vec{b} = \|\vec{v}\| \vec{u} \cdot \vec{b} \).

But \( \|\vec{v}\| \vec{v} \cdot \vec{b} = \|\vec{u}\| \|\vec{v}\|^2 \cos \phi - \|\vec{v}\| \vec{u} \cdot \vec{v} \)
and \( \|\vec{v}\| \vec{u} \cdot \vec{b} = \|\vec{u}\|^2 \|\vec{v}\| \cos \theta - \|\vec{v}\| \vec{u} \cdot \vec{v} \) \( \checkmark \).
2. (20pts) (i) Find the equation of the plane through the three points $P_0 = (1, 1, 2)$, $P_1 = (-1, 2, -2)$ and $P_2 = (2, -1, 1)$.

\[
\overrightarrow{P_0P_1} = (-2, 1, -4), \quad \overrightarrow{P_2P_0} = (1, -2, -1)
\]

\[
\overrightarrow{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_2P_0} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-2 & 1 & -4 \\
1 & -2 & -1
\end{vmatrix} = \begin{vmatrix}
1 & -4 \\
-2 & -1
\end{vmatrix} \hat{i} - \begin{vmatrix}
-2 & 1 \\
1 & -2
\end{vmatrix} \hat{j} + \begin{vmatrix}
-2 & 1 \\
1 & -2
\end{vmatrix} \hat{k} = -12 \hat{i} - 3 \hat{j} + 11 \hat{k}
\]

So \(3 \hat{i} + 2 \hat{j} - \hat{k}\) normal to plane.

\[3(x-1) + 2(y-1) - (z-2) = 0\] is the equation of the plane.

(ii) Find the distance between this plane and the point $Q = (1, 1, 1)$.

Let $R$ be the closest pt. Then $\overrightarrow{RQ}$ parallel to $Q = (3, 2, -1)$.

Method I $R$ lie on plane and line through $Q$ parallel to $(3, 2, -1)$.

\[
\overrightarrow{RQ} = t(3, 2, -1) - (x-1, y-1, z-1) = (3t, 2t, -t)
\]

\[
(x, y, z) = (3t + 1, 2t + 1, 1 - t)
\]

$R$ on plane

\[3(2t) + 2(2t) - (1 - t - 2) = 0\]

\[14t = -1 \quad t = -\frac{1}{14} \]

\[
R = \frac{1}{14}(11, 12, 6)
\]

Distance $= \frac{1}{14} \sqrt{114} = \frac{1}{\sqrt{14}}$.

\[
\overrightarrow{RQ} = \frac{1}{14}(3, 2, -1)
\]
Method II

\[\overrightarrow{RQ} = \text{proj}_{\overrightarrow{n}} \overrightarrow{P_0Q}\]

\[= \left( \frac{\overrightarrow{P_0Q} \cdot \overrightarrow{n}}{||\overrightarrow{n}||^2} \right) \overrightarrow{n}\]

\[= \frac{1}{14} (3,2,1)\]
3. (20pts) (i) What is the angle between the diagonal of a cube and one of the edges it meets?

Let the vertices of the cube be \((0,0,0)\), \((1,0,0)\), \((0,1,0)\), \((0,0,1)\), \((1,1,0)\), \((1,0,1)\), \((0,1,1)\), and \((1,1,1)\).

Suppose the diagonal is from \((0,0,0)\) to \((1,1,1)\). The associated vector is \((1,1,1)\). Meets edge \((0,0,0), (1,0,0)\).

\[
\cos \theta = \frac{(1,0,0) \cdot (1,1,1)}{1 \cdot \sqrt{3}} = \frac{1}{\sqrt{3}} \quad \theta = \frac{\pi}{6}
\]

(ii) Find the angle between the diagonal of a cube and the diagonal of one of its faces.

Take diagonal from \((0,0,0)\) to \((1,1,0)\).

So we want angle between \((1,1,1)\) and \((1,1,0)\).

\[
\cos \theta = \frac{(1,1,1) \cdot (1,1,0)}{\sqrt{2} \cdot \sqrt{3}} = \frac{2}{\sqrt{2} \cdot \sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}
\]

\[
\theta = \cos^{-1} \left( \frac{\sqrt{2}}{\sqrt{3}} \right)
\]
4. (20pts) Let $D$ be the region inside the paraboloid $a^2 z = x^2 + y^2$ and outside the sphere of radius $a$ centred at the origin.

(i) Describe the region $D$ in cylindrical coordinates.

Inside the paraboloid: $a^2 z = r^2$, $r^2 \leq a^2$

outside the sphere: $x^2 + y^2 + z^2 > a^2$

$r^2 + z^2 > a^2$

So $r^2 \leq a^2$, $r^2 + z^2 \leq a^2$

(ii) Describe the region $D$ in spherical coordinates.

Inside the paraboloid: $(r \cos \phi)^2 \leq a^2 r \sin \phi$

$r \cos \phi \leq a^2 \tan \phi$

outside the sphere: $r \geq a$.

So $r \geq a$, $r \cos \phi \leq a^2 \tan \phi$. 

\[ 4 \]
5. (20pts) Determine whether or not the following limits exist, and if they do exist, then find the limit. Explain your answer.

(i) \[
\lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2}
\]

Yes, the limit does exist.

\[
\frac{x^4 - y^4}{x^2 + y^2} = x^2 - y^2 \quad (x, y) \neq (0,0)
\]

So \[
\lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} x^2 - y^2 = 0.
\]

(ii) \[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}
\]

No, the limit does not exist.

If we approach along line \( x = 0 \)

\[
\lim_{y \to 0} \frac{0}{y} = 0.
\]

If we approach along line \( y = x \)

\[
\lim_{x \to 0} \frac{x^2}{2x^2} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \neq 0.
\]