1. (18pts) (i) Give the definition of independence.

The vectors $v_1, v_2, \dots, v_n \in V$ are **independent** if whenever
$$0 = \sum_i r_i v_i,$$
for scalars $r_1, r_2, \dots, r_n$ then $r_i = 0$ for every i.

(ii) Give the definition of the dimension of a vector space.

V has **dimension** n if there is a basis with n elements.

(iii) Give the definition of the column space of a matrix.

If $A \in M_{n,m}(F)$ then the **column space** of A is the span of the columns of A.

(iv) Give the definition of a linear transformation.

A function $\phi\colon V \longrightarrow W$ is a **linear transformation** if $\phi(v_1 + v_2) = \phi(v_1)+\phi(v_2)$ and $\phi(rv) = r\phi(v)$ for every scalar r and v_1, v_2 and $v \in V$.

(v) Give the definition of the rank of a linear transformation.

The **rank** of the linear transformation $\phi\colon V \longrightarrow W$ is the dimension of the image of ϕ.

(vi) Give the definition of an eigenvector of a linear transformation.

An **eigenvector** of the linear transformation $\phi\colon V \longrightarrow V$ is a non-zero vector $v \in V$ such that $\phi(v) = \lambda v$ for some scalar λ.

2. (15pts) Let $\phi\colon P_d(\mathbb{R}) \longrightarrow P_{d+1}(\mathbb{R})$ be the function $\phi(f) = f' + (1 + x)f$, where f' is the derivative of f.
(i) Show that ϕ is linear.

The sum of two linear functions is linear and so it suffices to show that the two functions $f \longrightarrow f'$ and $f \longrightarrow (1 + x)f$ are linear. Standard properties of the derivative imply that the first function is linear. The second function is a sum of the two functions $f \longrightarrow f$ and $f \longrightarrow xf$. The first function is the identity and this is linear. The second function multiplies a polynomial by x. Suppose f_1 and f_2 are two polynomials.

$$x(f_1 + f_2) = xf_1 + xf_2,$$

so that multiplication by x respects addition. Suppose that f is a polynomial and r is a scalar

$$x(rf) = r(xf),$$

so that multiplication by x respects scalar multiplication. Thus ϕ is linear.

(ii) Suppose that $d = 3$. Find the matrix of ϕ with respect to the standard basis of $P_3(\mathbb{R})$.

The standard basis is 1, x, x^2 and x^3.

$\phi(1) = 1+x \quad \phi(x) = 1+x+x^2 \quad \phi(x^2) = 2x+x^2+x^3 \quad \phi(x^3) = 3x^2+x^3+x^4.$

The matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. (15pts) Let $\phi\colon F^4 \longrightarrow F^4$ be the function
$(w, x, y, z) \longrightarrow (w-2x-2y+z, 2w-4x-4y+2z, w-x+y-z, 2w-3x-y)$.
(i) Show that ϕ is linear.

Let

$$A = \begin{pmatrix} 1 & -2 & -2 & 1 \\ 2 & -4 & -4 & 2 \\ 1 & -1 & 1 & -1 \\ 2 & -3 & -1 & 0 \end{pmatrix}$$

Then $\phi(v) = Av$ and so ϕ is linear.

(ii) Find a basis for the kernel of ϕ. What is the nullity of ϕ?

We apply Gaussian elimination to A. We add -2, -1 and -2 times the first row to the second, third and fourth rows to get

$$\begin{pmatrix} 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{pmatrix}.$$

Subtracting the third row from the fourth row and switching rows we get

$$\begin{pmatrix} 1 & -2 & -2 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We apply back substitution to find the kernel. z and y are free variables and so the nullity is two. Using the second equation we get $x = -3y+2z$ and using the first equation we get $w - 2(-3y + 2z) - 2y + z = 0$, so that $w = -4y + 3z$. The general element of the kernel is

$$(-4y + 3z, -3y + 2z, y, z) = y(-4, -3, 1, 0) + z(3, 2, 0, 1).$$

Thus $(-4, -3, 1, 0)$ and $(3, 2, 0, 1)$ span the kernel. Since they are not parallel vectors they are independent and so they form a basis of the kernel.

(iii) Find a basis for the image of ϕ. What is the rank of ϕ?

The rank of ϕ is the number of pivots, which is two. The column space is equal to the rank. So we are looking for two independent column vectors. As $(1, 2, 1, 2)$ and $(-2, -4, -1, -3)$ are not parallel they are independent and so they are a basis for the image of ϕ.

4. (10pts) If $A \in M_{2,2}(\mathbb{R})$ is the matrix

$$\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$$

then find a closed form expression for A^n.

We first diagonalise A. The characteristic polynomial is

$$(1-\lambda)(4-\lambda) - 10 = 0.$$

Rearranging gives

$$\lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1) = 0.$$

Thus the eigenvalues are $\lambda = 6$ and $\lambda = -1$. If we plug in $\lambda = 6$ into the matrix $B = A - \lambda I_2$ then we get

$$\begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix}$$

So B is a matrix of rank one and the kernel is spanned by $(2, 5)$, which is an eigenvector of A with eigenvalue 6. If we plug in $\lambda = -1$ then we get

$$\begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}$$

The kernel of this matrix is spanned by $(1, -1)$ and this is an eigenvector with eigenvalue -1. Let

$$D = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$$

By general theory $A = PDP^{-1}$, where

$$P^{-1} = \frac{-1}{7} \begin{pmatrix} -1 & -1 \\ -5 & 2 \end{pmatrix}.$$

But then $A = PD^nP^{-1}$.

5. (30pts) For each statement below, say whether the statement is true or false. If it is false, give a counterexample and if it is true then explain why it is true.
(i) Every matrix $A \in M_{2,2}(\mathbb{R})$ is diagonalisable.

False. Let
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
The characteristic polynomial is $\lambda^2 = 0$ so that $\lambda = 0$ is the only eigenvalue. The kernel of this matrix is spanned by $(1, 0)$ so that the span of the eigenvalues is 1 dimensional. But if A is diagonalisable it would have a basis of eigenvectors.

(ii) Every function $\phi \colon F^n \longrightarrow F^m$ is linear, where F is a field.

False. The function $\phi \mathcal{Q} \longrightarrow \mathcal{Q}$ given by $x \longrightarrow x^2$ is not linear as $4 = \phi(2) \neq 1 + 1 = \phi(1) + \phi(1)$, so that ϕ does not respect addition

(iii) Two finite dimensional vector spaces of the same dimension over the same field F are isomorphic.

True. Suppose that V and W both have dimension n. Let $v_1, v_2, \dots, v_n$ be basis of V and let $w_1, w_2, \dots, w_n$ be a basis of W. Define a function $\phi \colon V \longrightarrow W$ as follows. Given $v \in V$ we may find unique scalars $r_1, r_2, \dots, r_n$ such that $v = \sum_i r_i v_i$. We define $\phi(v) = \sum_i r_i w_i$. ϕ is well-defined and a bijection. Suppose that p and $q \in V$. We may find scalars $r_1, r_2, \dots, r_n$ and $s_1, s_2, \dots, s_n$ such that $p = \sum_i r_i v_i$ and $q = \sum_i s_i v_i$. But then $p + q = \sum_i (r_i + s_i) v_i$ and so $\phi(p + q) = \sum_i (r_i + s_i) w_i = \sum_i i w_i + \sum_i s_i w_i = \phi(p) + \phi(q)$. Thus ϕ respects addition. Now suppose $\lambda \in F$. Then $\lambda p = \sum (\lambda r_i) v_i$. Thus $\phi(\lambda p) = \sum_i (\lambda r_i) w_i = \lambda (\sum_i r_i w_i) = \lambda \phi(p)$. Hence ϕ respects scalar multiplication. Thus ϕ is a linear isomorphism.

(iv) If a matrix $A \in M_{2,2}(F)$ has only one eigenvalue then A is not diagonalisable.

False. The matrix
$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
has characteristic polynomial $\lambda^2 = 0$. Thus $\lambda = 0$ is the only eigenvalue. But A is diagonal to start with, so A is diagonalisable.

(v) If a matrix $A \in M_{n,n}(F)$ has n distinct eigenvalues then A is diagonalisable.

True. It is enough to show that A has a basis of eigenvetors. Suppose that $v_1, v_2, \dots, v_n$ are the eigenvectors, with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. It is enough to show that these vectors are independent. Suppose not. Then we can find scalars $r_1, r_2, \dots, r_n$, not all zero, such that $\sum_i r_i v_i = 0$. We may assume that n is minimal with this property. Apply A to both sides we get $\sum_i r_i \lambda_i v_i = 0$. Multiply the first equation by λ_n and subtract to get $\sum_i (\lambda_n - \lambda_i) r_i v_i = 0$. As the last term is zero but the rest are non-zero this contradicts our choice of n. Thus $v_1, v_2, \dots, v_n$ are independent. Therefore they are a basis and A is diagonalisable.

(vi) Suppose that $A \in M_{n,n}(F)$ has n distinct eigenvalues and $B \in M_{n,n}(F)$ commutes with A. Then B is diagonalisable.

True. Let v be an eigenvector of A with eigenvalue λ. By assumption $E_\lambda(A)$ is spanned by v. Let $w = Bv$. We have
$$Aw = A(Bv) = (AB)v = (BA)v = B(Av) = B(\lambda v) = \lambda Bv = \lambda w.$$
Thus $w \in E_\lambda(A)$ and so $w = \mu v$ for some μ. But then v is an eigenvector with eigenvalue μ for B.
Let $v_1, v_2, \dots, v_n$ be eigenvectors of A. As $v_1, v_2, \dots, v_n$ have distinct eigenvalues, $v_1, v_2, \dots, v_n$ are a basis of F^n. But then $v_1, v_2, \dots, v_n$ are a basis of eigenvalues of B. Hence B is diagonalisable.