MODEL ANSWERS TO THE FIRST QUIZ

1. (18pts) (i) Give the definition of a $m \times n$ matrix.

A $m \times n$ **matrix** with entries in a field F is a function

$$A \colon I \times J \longrightarrow F,$$

where I is the set of integers between 1 and m and J is the set of integers from 1 to n.

(ii) Give the definition of row echelon form.

A matrix U is in **echelon form** if

- the first non-zero entry of every row, called a pivot, is 1.
- a row with a pivot is above a row without one.
- a pivot which comes above another pivot also occurs to the left of that pivot.

(iii) Give the definition of the rank of a matrix.

The **rank** of a matrix A is the number of pivots in a matrix U in row echelon form, which is also row equivalent to A.

(iv) Give the definition of the left inverse of a matrix.

If A is a $m \times n$ matrix then a **left inverse** of A is a $n \times m$ matrix B such that $BA = I_n$.

(v) Give the definition of linear combination.

A vector $v \in V$ is a **linear combination** of $v_1, v_2, \ldots, v_k \in V$ if there are scalars $r_1, r_2, \ldots, r_k$ such that $v = r_1v_1 + r_2v_2 + \ldots r_kv_k$.

(vi) Give the definition of closed under addition.

Let V be a vector space. We say that a subset $W \subset V$ is **closed under addition** if whenever v and w belong to W then so does their sum $v + w$.

2. (15pts) (i) Let

$$V = \{\, f \in P_d(\mathbb{R}) \mid f'(0) = 0 \,\},$$

be the set of real polynomials of degree at most d whose derivative at zero is zero. Is V a subspace of $P_d(\mathbb{R})$?

True. It suffices to check that V is non-empty, closed under addition and scalar multiplication. The zero polynomial has zero derivative at zero, and so V is certainly non-empty. If f and g are two elements of V then $f'(0) = g'(0) = 0$. But then

$$(f + g)'(0) = f'(0) + g'(0) = 0,$$

using the standard rules for differentation. It follows that $f + g \in V$ and so V is closed under addition. Finally if $f \in V$ and $\lambda \in \mathbb{R}$ then $f'(0) = 0$. But then

$$(\lambda f)'(0) = \lambda f'(0) = 0,$$

so that $\lambda f \in V$. Hence V is closed under scalar multiplication.

(ii) Let F be a field and let

$$V = \{\, A \in M_{2,2}(F) \mid \operatorname{rk}(A) \leq 1 \,\},$$

be the set of 2×2 matrices of rank at most one. Is V a subspace of $M_{2,2}(F)$?

False. We have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both matrices on the left have rank one but the matrix on the right has rank two. Therefore V is not closed under addition and so it cannot be a subspace.

3. (10pts) Under what conditions on the real number c is the vector $(1, c, 1) \in \mathbb{R}^3$ a linear combination of the vectors $v_1 = (-1, 3, 2)$ and $v_2 = (2, -6, -5) \in \mathbb{R}^3$?

Let A be the matrix whose columns are the vectors v_1 and v_2. Then

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -6 \\ 2 & -5 \end{pmatrix}.$$

Then v is a linear combination of v_1 and v_2 if and only if the equation $Ax = b$ has a solution. To determine this form the augmented matrix

$$\left(\begin{array}{cc|c} -1 & 2 & 1 \\ 3 & -6 & c \\ 2 & -5 & 1 \end{array}\right),$$

and apply Gaussian elimination. Multiply the first row by -1.

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 3 & -6 & 1 \\ 2 & -5 & c \end{array}\right).$$

Multiply the first row by -3 and -2 and add it the second and third rows

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & c+3 \\ 0 & -1 & 3 \end{array}\right).$$

Finally swap the second and third rows and multiply the second row by -1,

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & c+3 \end{array}\right).$$

The system of equations represented by this matrix has the same solutions as the system represented by the original matrix. The last equation can only be solved if $c + 3 = 0$ that is $c = -3$. In this case it is easy to solve these equations using back substitution. (As it happens $r_2 = -3$ and $r_1 = -7$ so that $(1, -3, 1) = -7(-1, 3, 2) - 3(2, -6, -5)$).

The span of v_1 and v_2 is a plane. Vectors of the form $(1, c, 1)$ form a line. The line and the plane intersect in a point (namely $(1, -3, 1)$), since the line is neither contained in the plane nor parallel to it.

4. (25pts) Let A be the real 3×4 matrix

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ -2 & 4 & -6 & 2 \\ 5 & -10 & 16 & 4 \end{pmatrix}.$$

(i) Express the matrix A as a product $A = PLU$ of a permutation matrix P, a lower triangular matrix L and a matrix U in echelon form.

We apply a modified version of Gaussian elimination. We recognise that the second row is a multiple of the first. So we first swap the second and third rows. This is the only permutation of the rows we need, so we have

$$P = P_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

After swapping the second and third rows of A we have

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 5 & -10 & 16 & 4 \\ -2 & 4 & -6 & 2 \end{pmatrix}.$$

Multiply the first row of A by -5 and 2 and add it to the second and third row

$$U = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As this completes the elimination, U is the indicated matrix. Finally,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

(ii) Find the kernel of A, that is the set of solutions to the homogeneous equation $Ax = 0$.

This is the same as the kernel of U. We solve the system $Ux = 0$ by back substitution. If the variables as (a, b, c, d) then b and d are free variables. The second equation determines c in terms of d, $c - d = 0$, that is $c = d$. The first equation then reads $a + 2b + 3d - d = 0$, so that $a = -2b - 2d$. Therefore

$$(-2b - 2d, b, d, d) = b(-2, 1, 0, 0) + d(-2, 0, 1, 1),$$

is the general solution to the homogeneous equation.

(iii) Given that $v = (1, 1, 1, 1)$ is a solution of the equation $Ax = b$, where $b = (1, -2, 15)$, what is the general solution?

$$(1, 1, 1, 1) + b(-2, 1, 0, 0) + d(-2, 0, 1, 1).$$

5. (30pts) For each statement below, say whether the statement is true or false. If it is false, give a counterexample and if it is true then explain why it is true. Let A be $m \times n$ matrix with entries in a field.

(i) If A $m < n$ then the equation $Ax = b$ always has a solution.

False. Let $A = (0, 0)$, $b = (1)$. Then $m = 1 < 2 = n$ and it is clear that the equation $Ax = b$ has no solutions.

(ii) Let $\phi\colon \mathbb{F}^n \longrightarrow F^m$ be the function $\phi(v) = Av$. If ϕ is surjective then the equation $Ax = b$ always has a solution.

True. Pick $b \in F^m$. As ϕ is surjective there is a vector $v \in F^n$ such that $\phi(v) = b$. Then $Av = \phi(v) = b$, so that $v \in F^n$ is a solution of the equation $Ax = b$.

(iii) If B is an invertible $n \times n$ matrix then the nullspace of A and the nullspace of AB are the same.

False. Let $A = (1, 0)$ and let

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $B^2 = I_2$ so that B is invertible. The kernel of A is

$$\{\, v = (x, y) \in \mathbb{R}^2 \mid x = 0 \,\},$$

and the kernel of $AB = (0, 1)$ is

$$\{\, v = (x, y) \in \mathbb{R}^2 \mid y = 0 \,\},$$

which are clearly not equal.

(iv) If B is an invertible $m \times m$ matrix then the nullspace of A and the nullspace of BA are the same.

True. Suppose that $v \in \operatorname{Ker} A$. Then

$$(BA)v = B(Av) = B0 = 0.$$

Thus $v \in \operatorname{Ker} BA$ and so $\operatorname{Ker} A \subset \operatorname{Ker} BA$. Let C be inverse of B. By what we already just proved $\operatorname{Ker} BA \subset \operatorname{Ker} C(BA)$. But $C(BA) = (CB)A = A$. So $\operatorname{Ker} BA \subset \operatorname{Ker} A$. Since we have an inclusion either way, we have $\operatorname{Ker} A = \operatorname{Ker} BA$.

(v) A matrix A can have at most one inverse.

True. Let B and C be two inverses of A. We compute $B(AC)$. Since C is an inverse of A, $B(AC) = BI_n = B$. On the other hand, as matrix multiplication is associative we have $B(AC) = (BA)C$. As B is the inverse of A, $(BA)C = I_nC = C$.

(vi) Suppose that the entries of A are rational numbers. If the equation $Ax = b$ has a real solution $v \in \mathbb{R}^n$ then it has a rational solution $w \in \mathbb{Q}^n$.

True. Let $C = (A|b)$ be the augmented matrix. We apply Gaussian elimination. At the end we get $(U|c)$, row equivalent to A, where U is in echelon form. The equation $Ux = c$ has the same real and rational solutions as the equation $Ax = b$, and U is a matrix with rational entries. Since the equation $Ux = c$ has a real solution, it follows that every row of zeroes of U is also a row of zeroes of the augmented matrix $(U|c)$.

But then we could solve the equation $Ux = c$ using back substitution. If we pick rational numbers for the free variables then we get rational solutions.