MODEL ANSWERS TO THE NINTH HOMEWORK

1. Clearly the first thing is to subtract λI_3, to get the matrix

$$N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By direct computation, we see that $N^3 = 0$ but $N^2 \neq 0$. It follows that minimal polynomial of N is x^3, so that the minimal polynomial of the original matrix is $(x - \lambda)^3$. But the only matrix in Jordan canonical form with this minimal polynomial is

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The general case proceeds in the same way. We subtract λI_k. The resulting matrix N has the property that $N^k = 0$ but $N^{k-1} \neq 0$ (in fact this a matrix with a 1 in the top right hand corner and a zero elsewhere). This implies that the minimal polynomial of A is $(x - \lambda)^k$ and so the Jordan canonical form of A is $B_k(\lambda)$.

2. Suppose that $A = B_k(\lambda)$. Consider $N = A^t - \lambda I_k$. By direct computation $N^k = 0$ and $N^{k-1} \neq 0$ (in fact it is the matrix with a single non-zero entry, a 1 in the bottom left hand corner). It follows that the minimal polynomial of A^t is $(x - \lambda)^k$. The only matrix in Jordan canonical form with this minimal polynomial is $B_k(\lambda)$. It follows that there is a matrix $P = P_k(\lambda)$ such that $B_k(\lambda)^t = P B_k(\lambda) P^{-1}$.

Now we turn to the general case. Since A is a matrix in Jordan canonical form, A^t is a matrix in block form, with blocks $B_{k_i}(\lambda_i)^t$ on the main diagonal and zeroes everywhere else. Let P be the block matrix, with $P_{k_i}(\lambda_i)$ on the main diagonal and zeroes everywhere else. Then the inverse of P is the block matrix, with $P_{k_i}(\lambda_i)^{-1}$ on the main diagonal and zeroes everywhere else (indeed the identity matrix is unique and by direct computation this matrix multiplied by P is the identity matrix). Computing block by block then we see that $P A^t P^{-1} = A$, so that A is the Jordan canonical form for A^t.

3. We may as well suppose that B is in Jordan canonical form. Observe that if $B = B_k(0)$ then

$$\nu(B^l) = \begin{cases} l & l \leq k-1 \\ k & l \geq k. \end{cases}$$

In the general case, suppose that n_l is the number of Jordan block of size $l \times l$, $1 \leq l \leq k$. Then, multiplying out B^l block by block and using the formula above we see that

$$\nu(B^l) = \sum_{i=1}^{l} n_i i + \sum_{i>l} n_i l.$$

In particular

$$\nu(B^{l+1}) - \nu(B^l) = n_{l+2} + n_{l+3} + \dots.$$

4. (i) *False.* Consider the matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Both matrices are in Jordan canonical form. The first consists of two Jordan blocks of type $B_2(0)$. The second consists of three Jordan blocks, one of type $B_2(0)$, and two of type $B_1(0)$, which are not the same even up to re-ordering.
For both matrices $A_1^2 = A_2^2 = 0$ and yet $A_1 \neq 0$ and $A_2 \neq 0$. So both matrices have minimal polynomial x^2, but they don't have the same Jordan canonical form.
(ii) *False.* Consider the matrices A_1 and A_2 above. We have already seen that they have the same minimal polynomial. The characteristic polynomial is x^4 in both cases. On the other hand, they don't have the same Jordan canonical form.
(iii) *True.* We might as well suppose that A is a diagonal matrix. If the entries on the main diagonal are $\lambda_1, \lambda_2, \dots, \lambda_n$ then the characteristic polynomial of A is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n).$$

The roots of this polynomial determine $\lambda_1, \lambda_2, \dots, \lambda_n$, whence the Jordan canonical form.
(iv) *False.* Let $V = \mathbb{R}^1$ and let

$$P = \{\, x \in \mathbb{R} \,|\, x \geq 0 \,\}.$$

Then P contains 0, it is closed under addition, since the sum of two non-negative is non-negative, but it is not closed under scalar multiplication as $1 \in P$ but $(-1) \cdot 1 = -1 \notin P$.

5. If we add the second column to the first column, we get

$$\begin{vmatrix} 2 & -2 & 1 & -1 \\ 3 & -2 & 2 & 1 \\ 1 & -1 & 1 & 4 \\ 0 & 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 1 & -1 \\ 1 & -2 & 2 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 2 & -1 \end{vmatrix}.$$

Expanding about the first column gives,

$$-\begin{vmatrix} -2 & 1 & -1 \\ -1 & 1 & 4 \\ 0 & 2 & -1 \end{vmatrix}.$$

Expanding about the first column of this matrix gives,

$$2\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 2(-1-8) - (-1+2) = -18 - 1 = -19.$$

6. Let $v_n = (s_{n-1}, s_n, s_{n+1})$, so that $v_1 = (0, 0, 1)$, and let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}.$$

Then $Av_n = v_{n+1}$. It follows that $v_n = A^{n-1}v_1$, and so we need to compute powers of A. To this end, we need to diagonalise A. The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -2 & 2-\lambda \end{vmatrix} \\ &= -\lambda(-\lambda(2-\lambda) - 1) - 2 \\ &= -\lambda^3 + 2\lambda^2 + \lambda - 2. \end{aligned}$$

Thus we want to find the solutions to the equation

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0.$$

By inspection $\lambda = \pm 1$ are both solutions. If the roots of $x^3 + ax^2 + bx + c$ are α, β and γ then

$$x^3 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma),$$

and multiplying out we see that the product of the roots is $-c$. Thus the third root is 2. We calculate the eigenspaces in all three cases. For $\lambda = \lambda_1 = 1$ we want the kernel of

$$A - I_3 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}.$$

Multiplying the first row by -1 we get

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}.$$

Multiplying the first row by 2 and adding it to the third row we get

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Multiplying the second row by -1 and then adding it to the third row gives

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

a row of zeroes, as expected. z is a free variable. Put $z = 1$. Then $y = 1$ and $x = 1$, so that the kernel is spanned by $(1, 1, 1)$. Thus $(1, 1, 1)$ is an eigenvector with eigenvalue $\lambda_1 = 1$.
Now suppose $\lambda = \lambda_2 = -1$. Then we want the kernel of

$$A + I_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}.$$

Adding twice the first row to the third row gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix}.$$

Subtracting the second row from the third gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Once again z is a free variable. If we put $z = 1$, then $y = -1$ and $x = 1$. Thus $(1, -1, 1)$ is an eigenvector with eigenvalue $\lambda_2 = -1$.

Finally suppose that $\lambda = \lambda_3 = -2$. Then we want the kernel of

$$A - 2I_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

Subtracting the first row from the third row gives

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we put $z = 4$, then $y = 2$ and $x = 1$. Thus $(1, 2, 4)$ is an eigenvector with eigenvalue $\lambda_3 = 2$. It follows that $A = PDP^{-1}$, where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix}.$$

We compute the inverse of P, by using Gauss-Jordan elimination

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & -1 & 2 & | & 0 & 1 & 0 \\ 1 & 1 & 4 & | & 0 & 0 & 1 \end{pmatrix}.$$

We subtract the first row from the second and third rows to get

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & -2 & 1 & | & -1 & 1 & 0 \\ 0 & 0 & 3 & | & -1 & 0 & 1 \end{pmatrix}.$$

Now we multiply the second row by $-1/2$ and the third row by $1/3$ to get

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1/2 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & | & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Now we multiply the third row by $1/2$ and -1 and add it to the second and first rows to get

$$\begin{pmatrix} 1 & 1 & 0 & | & 4/3 & 0 & -1/3 \\ 0 & 1 & 0 & | & 1/3 & -1/2 & 1/6 \\ 0 & 0 & 1 & | & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Finally we multiply the second row by -1 and add it to the first row to get

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & | & 1/3 & -1/2 & 1/6 \\ 0 & 0 & 1 & | & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Therefore

$$P^{-1} = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1/3 & -1/2 & 1/6 \\ -1/3 & 0 & 1/3 \end{pmatrix}.$$

It follows that $A^n = PD^nP^{-1}$. Now $v_{n+1} = A^n v_1$. Note that we only want the first entry of v_{n+1}. Now

$$P^{-1}v_0 = (-1/2, 1/6, 1/3)^t.$$

Therefore

$$D^nP^{-1}v_0 = (-1/2, (-1)^n/6, 2^n/3)^t.$$

Thus the first entry of v_{n+1} is

$$s_n = -1/2 + (-1)^n/6 + 2^n/3 = \frac{(-1)^n + 2^{n+1} - 3}{6}.$$

which is a closed form expression for s_n.

7. First the diagonal matrices,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The minimal polynomials are x, $x(x-1)$, $x(x-1)$, $x(x-1)$ and $x-1$. The characteristic polynomials are x^4, $x^3(x-1)$, $x^2(x-1)^2$, $x(x-1)^3$ and $(x-1)^4$. Now the matrices with one Jordan block of size 2×2:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with minimal polynomials x^2, $x^2(x-1)$ and $x^2(x-1)$, and characteristic polynomials x^4, $x^3(x-1)$ and $x^2(x-1)^2$, and

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with minimal polynomials $(x-1)^2$, $(x-1)^2x$ and $(x-1)^2x$, and characteristic polynomials $(x-1)^2x^2$, $(x-1)^3x$ and $(x-1)^4$. There are the matrices with two Jordan blocks of type 2×2,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with minimal polynomials x^2, $x^2(x-1)^2$ and $(x-1)^2$, and characteristic polynomials x^4, $(x-1)^2x^2$ and $(x-1)^4$. There are the matrices with one Jordan block of size three,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with minimal polynomials x^3, $x^3(x-1)$, $(x-1)^3x$ and $(x-1)^3$, and characteristic polynomials x^4, $x^3(x-1)$, $(x-1)^3x$ and $(x-1)^4$. Finally there are two matrices with one Jordan block,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with minimal polynomials x^4 and $(x-1)^4$ and characteristic polynomials x^4 and $(x-1)^4$.

8. The determinant is

$$\prod_{i>j}(\lambda_i - \lambda_j).$$

Here is how to see this. Suppose that we turn λ_i into a variable x. If we expand the determinant about the ith column, then we clearly get a polynomial $f_i(x)$ of degree $n-1$ in x. On the other hand, if we set $x = \lambda_j$, then two columns of the matrix are duplicates and so the determinant is zero. This tells us that $x - \lambda_j$ is factor of $f_i(x)$.

Going back to the original problem, if we expand the determinant, we get a polynomial of degree $n-1$ in each variable. By what we just proved $\lambda_i - \lambda_j$ is a factor. If we expand

$$\mu\prod_{i>j}(\lambda_i - \lambda_j),$$

where μ is a scalar, then we get a polynomial of degree $n-1$ in each variable. The only thing that is left to determine is the factor μ. One can see that it is 1 by comparing coefficients of the same monomial, say the one that comes from the diagonal,

$$\lambda_2\lambda_3\dots\lambda_n^{n-1}.$$

For the determinant it is one. For the product, one needs to take the first term from every bracket, so that for the product it is μ. Thus $\mu = 1$.