MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. (i) Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -1 & 2 \\ 3 & 6 & -2 & 5 \end{pmatrix}.$$

Then $\phi(v) = Av$, where $v = (r_1, r_2, r_3, r_4)$. In particular ϕ is linear.
(ii) We apply Gauss-Jordan elimination to A. We multiply the first row by -2 and -3 and add it to the second and third rows to get

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{pmatrix}.$$

Now we multiply the second row by -1 and add it to the third row to get

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This completes the Gaussian elimination. Now add the second row to the first to get

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This competes the Gauss-Jordan elimination. The product of the corresponding elementary row matrices is

$$\begin{aligned} E &= E_{2,1}(1)E_{2,3}(-1)E_{1,3}(-3)E_{1,2}(-2) \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}. \end{aligned}$$

This defines a linear isomorphism

$$g \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3,$$

by the rule

$$g(s_1, s_2, s_3) = (-s_1 + s_2, -2s_1 + s_2, -0s_1 - s_2 + s_3).$$

To finish off, we need to apply Gaussian elimination to the tranpose matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{pmatrix}.$$

We multiply the first row by -2 and add it to the second and forth rows to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Now we swap the second and third rows and add the second row to the fourth row to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The product of the corresponding elementary row operations is then

$$\begin{aligned} E' &= E_{2,4}(1)P_{1,2}E_{1,4}(-2)E_{1,2}(-2) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The inverse matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 \end{pmatrix}.$$

The transpose of this matrix is

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This defines a linear map $f\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ by the rule

$$f(r_1, r_2, r_3, r_4) = (r_1 + 2r_2 + 2r_4, r_3, r_2 - r_4, r_4).$$

The corresponding map ψ, which is given by the transpose of the last 4×3 matrix above, is $\psi(r_1, r_2, r_3, r_4) = (r_1, r_2, 0)$.

2. (i) One direction is clear. If $f(x) = (x - \alpha)g(x)$ then $f(\alpha) = (\alpha - \alpha)g(\alpha) = 0$ so that α is a root of $f(x)$.

Now suppose that α is a root of $f(x)$. By the division algorithm

$$f(x) = (x - \alpha)g(x) + r(x),$$

where $r(x) = 0$ or $r(x)$ has degree zero. Either way $r(x) = r_0$ is a constant polynomial. If we plug in α then both $f(\alpha)$ and the term $(\alpha - \alpha)g(\alpha)$ are zero. But then $r_0 = 0$ as well. It follows that $x - \alpha$ is a factor of $f(x)$.

(ii) Suppose that $g(x) = g_m x^m + g_{m-1}x^{m-1} + \cdots + g_0$ and $h(x) = h_n x^n + h_{n-1}x^{n-1} + \cdots + h_0$, where g_m and $h_n \neq 0$.

$$f(x) = g_m h_n x^{n+m} + \cdots ,$$

where dots indicate lower terms. As F is a field, $g_m h_n \neq 0$. Then the degree of $g(x)$ is m, the degree of $h(x)$ is n and the degree of $f(x)$ is $n + m$, the sum of the degrees.

3. (i) Suppose that $f(x) = g(x)h(x)$. Then the degree of $f(x)$ is the sum of the degrees of $g(x)$ and $h(x)$. But if $f(x)$ has degree at most three and $g(x)$ and $h(x)$ have degree at least one then at least one of $g(x)$ and $h(x)$ has degree one. Possibly switching $g(x)$ and $h(x)$, we may assume that $g(x)$ has degree one. But then $g(x) = g_1 x + g_0$, where $g_1 \neq 0$. Let $g_1(x) = x + g_0/g_1$ and $h_1(x) = g_1 h(x)$. Then $f(x) = g_1(x)h_1(x)$. But $g_1(x) = x - \alpha$, where $\alpha = -g_0/g_1$.

(ii) A general monic degree two polynomial with coefficients in $\mathbb{F}_3$ looks like

$$f(x) = x^2 + ax + b,$$

where a and $b \in \{0, 1, 2\}$. Such a polynomial is irreducible, provided it has no roots. 0 is a root if and only if $b = 0$. So we may assume that $b = 1$ or 2. Suppose that $b = 1$. Then

$$f(1) = 1 + a + 1.$$

So 1 is a root if and only if $a = 1$.

$$f(2) = 4 + 2a + 1 = 2a + 2.$$

So 2 is a root if and only if $a = 2$. Thus

$$f(x) = x^2 + 1,$$

is irreducible. Now suppose that $b = 2$. Then

$$f(1) = 1 + a + 2,$$

so that 1 is a root if and only if $a = 0$.

$$f(2) = 4 + 2a + 2,$$

so that 2 is a root if and only if $a = 0$. Thus

$$f(x) = x^2 + x + 2 \qquad \text{and} \qquad f(x) = x^2 + 2x + 2,$$

are the only other irreducible monic polynomials of degree three.
(iii) A general degree three polynomial with coefficients in $\mathbb{F}_2$ looks like

$$f(x) = x^3 + ax + bx + c,$$

where a, b and c belong to $\{0, 1\}$. 0 is a root if and only if $c = 0$. So we may assume that $c = 1$.

$$f(1) = 1 + a + b + 1 = a + b.$$

So 1 is a root if and only if $a = b = 0$ or $a = b = 1$. Thus the irreducible polynomials of degree three are

$$x^3 + x^2 + 1 \qquad \text{and} \qquad x^3 + x + 1.$$

4. (i) *False.* $x^2 + 1$ is a real polynomial of degree two without any real roots. Therefore $x^2 + 1$ is irreducible.
(ii) *False.* $A = (0) \in M_{1,1}(F)$ is not invertible but it is diagonal and therefore clearly diagonalisable.
(iii) *True.* By assumption $B = PAP^{-1}$, for some invertible matrix $P \in M_{n,n}(F)$. On the other hand $A = QDQ^{-1}$, for some invertible matrix $Q \in M_{n,n}(F)$ and diagonal matrix $D \in M_{n,n}(F)$. Therefore

$$\begin{aligned} B &= PAP^{-1} \\ &= P(QDQ^{-1})P^{-1} \\ &= (PQ)D(P^{-1}Q^{-1}) \\ &= (PQ)D(PQ)^{-1} \\ &= RDR^{-1}, \end{aligned}$$

where $R = PQ$. So B is diagonalisable.
(iv) *False.* Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then A is a real 2×2 matrix. The characteristic equation is

$$\lambda^2 + 1 = 0.$$

This has no real roots and so A is not diagonalisable as a real matrix. It has two distinct complex roots $\pm i$ and so it is diagonalisable as a complex matrix.

5. The general degree four polynomial looks like

$$f(x) = x^4 + ax^3 + bx^2 + cx + d.$$

Now this has a linear factor if and only if it has a root. 0 is a root if and only if $d = 0$. So we may assume that $d = 1$. In this case 1 is a root if and only if $a + b + c = 0$. Let z be the number of a, b and c equal to zero. Then we may assume that z is zero or two.

The remaining possibility is if $f(x)$ factors as a product of two quadratics $g(x)$ and $h(x)$. Now we may assume that both of these quadratics are irreducible (else $f(x)$ has a linear factor, whence it has a root, which we have already eliminated).

Now the general quadratic looks like

$$x^2 + ex + f.$$

0 is a root if and only if $f = 0$. So we may assume that $f = 1$. Then 1 is a root if and only if

$$1 + e + 1 = 0,$$

that is $e = 0$. So $x^2 + x + 1$ is the unique irreducible quadratic. Its square is

$$x^4 + x^2 + 1.$$

(Note that $(a + b)^2 = a^2 + b^2$ over $\mathbb{F}_2$!). Thus

$$x^4 + x + 1, \qquad x^4 + x^3 + 1 \qquad \text{and} \qquad x^4 + x^3 + x^2 + x + 1,$$

are the irreducible quartics with coefficients in $\mathbb{F}_2$.