MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. By assumption there is an invertible matrix P such that $A = PBP^{-1}$. Let $\lambda \in F$.

Claim 0.1.
$$E_\lambda(B) = P(E_\lambda(A)).$$

Proof. Pick $v \in E_\lambda(B)$. Then $Bv = \lambda v$ and so
$$\begin{aligned} A(Pv) &= (PBP^{-1})Pv \\ &= PB(P^{-1}P)v \\ &= PBv \\ &= P(\lambda v) \\ &= \lambda(Pv). \end{aligned}$$
Thus $Pv \in E_\lambda(A)$. It follows that
$$E_\lambda(B) \subset P(E_\lambda(A)).$$
Let $Q = P^{-1}$. Then $B = QAQ^{-1}$ and so by what we have already proved,
$$E_\lambda(A) \subset Q(E_\lambda(B)).$$
Multiply both sides by P (here we mix up slightly the difference between multiplication by P and applying the linear function $\psi \colon V \longrightarrow V$ given by P), to get
$$P(E_\lambda(A)) \subset P(Q(E_\lambda(B))) = (P \circ Q)(E_\lambda(B)) = E_\lambda(B).$$
This gives the reverse inclusion and establishes the claim. $\square$

Since P is invertible, it follows that
$$\dim E_\lambda(B) = \dim E_\lambda(A).$$
The matrices
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
are similar. Indeed, just take
$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
$E_0(A)$ is spanned by $(1,0)$ and $E_0(B)$ is spanned by $(0,1)$. So the two eigenspaces are not equal, even though they have the same dimension.

2. Denote by A the first matrix and by B the second matrix.
(a) $E_1(A)$ span of $(1, 0)$ whilst $E_1(B) = F^2$. Thus

$$\dim E_1(A) = 1 \neq 2 = \dim E_1(B).$$

Thus A and B are not similar.
(b) $E_0(A) = \{0\}$ whilst $E_0(B)$ is the span of $(1, 0)$. Thus

$$\dim E_0(A) = 0 \neq 1 = \dim E_0(B).$$

Thus A and B are not similar.
(c) Let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then P is its own inverse and $A = PBP^{-1}$. So A and B are similar.
3. Let $v_1, v_2, \dots, v_n$ be a basis of eigenvectors for A. Since A and B have the same eigenspaces, A and B have the same eigenvectors. But then $v_1, v_2, \dots, v_n$ is a basis of eigenvectors for B. Let P be the matrix whose columns are the vectors $v_1, v_2, \dots, v_n$. We proved in class that $A = PDP^{-1}$ and $B = PEP^{-1}$, where D and E are diagonal matrices. We have

$$\begin{aligned} AB &= (PDP^{-1})(PEP^{-1}) \\ &= (PD)(P^{-1}P)(EP^{-1}) \\ &= P(DE)P^{-1}. \end{aligned}$$

Similarly (or by symmetry)

$$BA = P(ED)P^{-1}.$$

But $DE = ED$ as D and E are diagonal matrices and so $AB = BA$.
4. Let $v \in W$. If $v = 0$ then $T(v) = 0 \in W$. Otherwise we can find scalars $r_1, r_2, \dots, r_k$ and eigenvectors $v_1, v_2, \dots, v_k$ such that

$$v = \sum_i r_i v_i.$$

Suppose that v_i is an eigenvector with eigenvalue λ_i. Then

$$\begin{aligned} T(v) &= T(\sum_i r_i v_i) \\ &= \sum r_i T(v_i) \\ &= \sum r_i \lambda_i v_i \\ &= \sum s_i v_i, \end{aligned}$$

where $s_i = r_i \lambda_i$. But then $T(v) \in W$ and so $T(W) \subset W$.

5. Let $v_1, v_2, \dots, v_{n-1}$ be a basis for the kernel and let v_n be an eigenvector with non-zero eigenvalue $\lambda \neq 0$. Then $T(v_n) = \lambda v_n \neq 0$ so v_n does not belong to the kernel of T. In particular v_n is not in the span of $v_1, v_2, \dots, v_{n-1}$ and so $v_1, v_2, \dots, v_n$ are independent. As V has dimension n it follows that $v_1, v_2, \dots, v_n$ is a basis of V. But then T is diagonalisable.

6. As T is diagonalisable, V has a basis of eigenvectors $v_1, v_2, \dots, v_m$ for T. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_m$ are the eigenvalues. Then

$$T^n(v_i) = \lambda_i^n v_i.$$

So v_i is an eigenvector with eigenvalue λ_i^n. Therefore $v_1, v_2, \dots, v_m$ is a basis of eigenvectors for T^n.