MODEL ANSWERS TO THE SIXTH HOMEWORK

1. Let $n = \dim V$. If we apply rank-nullity to T and $T \circ T$ we get

$$\nu(T) + \mathrm{rk}(T) = n \qquad \text{and} \qquad \nu(T \circ T) + \mathrm{rk}(T \circ T) = n.$$

Subtracting one equation from the other, we get $\nu(T) = \nu(T \circ T)$. Suppose that $v \in \operatorname{Ker} T$. Then $T(v) = 0$ and so certainly $(T \circ T)(v) = T(T(v)) = 0$. But then $v \in \operatorname{Ker}(T \circ T)$. Thus $\operatorname{Ker} T \subset \operatorname{Ker}(T \circ T)$.
But suppose that $U \subset W \subset V$ are two linear subspaces of the same dimension. Start with a basis of U and extend it to a basis of W. Since the subspaces are of the same dimension it follows that we could not have added any vectors and so $U = W$.
As both $\operatorname{Ker}(T)$ and $\operatorname{Ker}(T \circ T)$ have the same dimension, it follows that we have equality, $\operatorname{Ker} T = \operatorname{Ker}(T \circ T)$. Suppose $v \in \operatorname{Ker} T \cap \operatorname{Im} T$. Then $v = T(w)$ and so $(T \circ T)(w) = T(T(w)) = T(v) = 0$. But then $w \in \operatorname{Ker}(T \circ T)$. By what we already just proved, this means $w \in \operatorname{Ker} T$. But then $v = T(w) = 0$. Therefore $\operatorname{Ker} T \cap \operatorname{Im} T = \{0\}$.
2. By assumption $\operatorname{Ker} T = \operatorname{Im} T$, so that $\nu(T) = \mathrm{rk}(T) = r$, say. By rank-nullity,

$$n = \nu(T) + \mathrm{rk}(T) = 2r,$$

is even.
3. (a) Suppose that u and $v \in V$. Then

$$\begin{aligned}(R + S)(u + v) &= R(u + v) + S(u + v)\\ &= Ru + Rv + Su + Sv\\ &= Ru + Su + Rv + Sv\\ &= (R + S)u + (R + S)v.\end{aligned}$$

Thus $R + S$ respects addition. Now suppose that $v \in V$ and $\lambda \in F$. Then

$$\begin{aligned}(R + S)(\lambda v) &= R(\lambda v) + S(\lambda v)\\ &= \lambda R(v) + \lambda S(v)\\ &= \lambda(R(v) + S(v))\\ &= \lambda((R + S)v)).\end{aligned}$$

Thus $R + S$ respects scalar multiplication. It follows that $R + S$ is linear.

(b) Suppose that $w \in \operatorname{Im}(R+S)$. Then $w = (R+S)(v) = Rv + Sv$. Thus

$$\operatorname{Im}(R+S) \subset \operatorname{Im} R + \operatorname{Im} S.$$

We have already seen that

$$\dim(\operatorname{Im} R + \operatorname{Im} S) = \dim(\operatorname{Im} R) + \dim(\operatorname{Im} S) - \dim(\operatorname{Im} R \cap \operatorname{Im} S) \leq \operatorname{rk}(R) + \operatorname{rk}(S).$$

Thus

$$\operatorname{rk}(R+S) \leq \operatorname{rk}(R) + \operatorname{rk}(S).$$

4. The standard basis is 1, t, t^2 and t^3. We have

$$\operatorname{Int}(t^i) = \int_0^x t^i \, dt = \frac{x^{i+1}}{i+1}.$$

Thus the matrix representation is $\operatorname{Int}\colon P_3(\mathbb{R}) \longrightarrow P_4(\mathbb{R})$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

5. Complex conjugation sends 1 to 1 and i to $-i$. Thus the matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

6. We first check that the function ϕ is linear. This is straightforward. Now consider the equation

$$\phi(fg) = f\phi(g) + \phi(f)g.$$

Suppose that we fix f and vary g. Since multiplication by g is linear and the composition of linear functions is linear, both sides are linear in g. As the polynomials 1, x, x^2, ..., x^n, ..., span the vector space of all polynomials, we may assume that $g = x^e$. Similarly if we then fix g and vary f, we get a linear function of f. So we may assume that $f = x^d$.

We now compute the LHS. The product is x^{d+e} and

$$\phi(x^{d+e}) = (d+e)x^{d+e-1}.$$

Let us turn to the RHS.

$$x^d\phi(x^e) + \phi(x^d)x^e = x^d e x^{e-1} + d x^{d-1} x^e = (d+e)x^{d+e-1}.$$

Thus both sides are the same.