MODEL ANSWERS TO THE THIRD HOMEWORK

1. (i) We apply Gaussian elimination to A. First note that the second row is a multiple of the first row. So we need to swap the second and third rows.

$$\begin{pmatrix} -1 & 3 & -2 & 1 \\ 2 & -6 & 5 & -7 \\ -3 & 9 & -6 & 3 \end{pmatrix}.$$

Since these are the only rows which will need to be swapped, we already know

$$P = P_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We multiply the first row by -1,

$$\begin{pmatrix} 1 & -3 & 2 & -1 \\ 2 & -6 & 5 & -7 \\ -3 & 9 & -6 & 3 \end{pmatrix}.$$

We multiply the first row by -2 and 3 and add it to the second and third rows,

$$U = \begin{pmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since this matrix is in echelon form we know that this is the matrix U. The matrix L is a record of the other steps of Gaussian elimination,

$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.$$

(ii) Two, since U contains two pivots.
(iii) Clearly this is the same as solving the system $Ux = 0$. Suppose the variables are x_1, x_2, x_3 and x_4. Then x_2 and x_4 are free variables. We pick any value for these variables and use those values to determine the values for x_3 and x_1 using back substitution. We can use the second equation to determine the value for x_3, in terms of x_4,

$$x_3 - 5x_4 = 0,$$

so that $x_3 = 5x_4$. Now use the first equation to determine x_1 in terms of x_2 and x_4,

$$x_1 - 3x_2 + 2(5x_4) - x_4 = 0,$$

so that $x_1 = 3x_2 - 9x_4$. The general solution to the homogeneous is

$$(x_1, x_2, x_3, x_4) = (3x_2 - 9x_4, x_2, 5x_4, x_4) = x_2(3, 1, 0, 0) + x_4(-9, 0, 5, 1).$$

(iv) We could solve this equation in the usual fashion, but here is a slick trick. We want to solve

$$PLUx = b.$$

Multiply both sides by P (which is its own inverse) to get

$$LUx = Pb.$$

Let $y = Ux$. Then we need to solve

$$Ly = Pb \qquad \text{and} \qquad Ux = y.$$

The trick is that both systems of equations can be solved very quickly. The first by forward substitution and the second by back substitution.

$$Pb = \begin{pmatrix} 1 \\ -3 \\ -6 \end{pmatrix}.$$

The first equation for $Ly = Pb$ then reads

$$-y_1 = 1,$$

so that $y_1 = -1$. We use this value of y_1 in the second equation to determine y_2,

$$-2 + y_2 = -6,$$

so that $y_2 = -4$. Finally we use these values of y_1 and y_2 in the third equation to determine y_3,

$$3 + y_3 = 3,$$

so that $y_3 = 0$. Now we solve the system $Ux = y$. Note that these equations are consistent as $y_3 = 0$. As before x_2 and x_4 are free variables. We can use the second equation

$$x_3 - 5x_4 = -4,$$

to determine $x_3 = -4 + 5x_4$ in terms of x_4. We then use the first equation to determine x_1 in terms of x_2 and x_4,

$$x_1 - 3x_2 + 2(-4 + 5x_4) - x_4 = -1,$$

so that $x_1 = 3x_2 - 9x_4 + 7$. Therefore

$$(x_1, x_2, x_3, x_4) = (3x_2 - 9x_4 + 7, x_2, 5x_4 - 4, x_4) = x_2(3, 1, 0, 0) + x_4(-9, 0, 5, 1) + (7, 0, -4, 0).$$

Note that a solution to the second equation is the same as a solution to the first equation plus a particular solution, $(7, 0, -4, 0)$ to the second equation.

2. We first form the super augmented matrix:

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 0 & | & 0 & 1 & 0 \\ 1 & 2 & 2 & | & 0 & 0 & 1 \end{pmatrix}.$$

Now we apply Gaussian elimination. We multiply the first row by -2 and -1 and add it to the second and third rows to get

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & -1 & -2 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{pmatrix}.$$

Now we multiply the second row by -1 to get

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 2 & -1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{pmatrix}.$$

This completes Gaussian elimination. Now we continue with Gauss Jordan elimination. We multiply the third row by -2 and -1 and add it to the second and first rows to get

$$\begin{pmatrix} 1 & 2 & 0 & | & 2 & 0 & -1 \\ 0 & 1 & 0 & | & 4 & -1 & -2 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{pmatrix}.$$

Finally we multiply the second row by -2 and add it to the first row to get

$$\begin{pmatrix} 1 & 0 & 0 & | & -6 & 2 & 3 \\ 0 & 1 & 0 & | & 4 & -1 & -2 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{pmatrix}.$$

So the inverse matrix is

$$\begin{pmatrix} -6 & 2 & 3 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}.$$

3. We first form the super augmented matrix:

$$\begin{pmatrix} a & b & | & 1 & 0 \\ c & d & | & 0 & 1 \end{pmatrix}.$$

We apply Gaussian elimination. We multiply the first row by $1/a$, to get

$$\begin{pmatrix} 1 & b/a & | & 1/a & 0 \\ c & d & | & 0 & 1 \end{pmatrix}.$$

Now multiply the first row by $-c$ and add it to the second row,

$$\begin{pmatrix} 1 & b/a & | & 1/a & 0 \\ 0 & (ad-bc)/a & | & -c/a & 1 \end{pmatrix}.$$

Now multiply the second row by $a/(ad-bc)$ to get

$$\begin{pmatrix} 1 & b/a & | & 1/a & 0 \\ 0 & 1 & | & -c/(ad-bc) & a/(ad-bc) \end{pmatrix}.$$

Finally multiply the second row by $-b/a$ and add it to the first row,

$$\begin{pmatrix} 1 & 0 & | & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & | & -c/(ad-bc) & a/(ad-bc) \end{pmatrix}.$$

So we guess that A is invertible if and only if $ad - bc \neq 0$ and in this case we guess the inverse matrix is

$$B = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

By direct computation we see that $BA = AB = I_2$, so B is the inverse of A, provided $ad \neq bc$. Now suppose that $ad = bc$. Suppose that $bd \neq 0$. If we multiply the first row by d and the second row by b then we get

$$\begin{pmatrix} ad & bd \\ bc = ad & bd \end{pmatrix}$$

If we then multiply the first row by -1 and add it the second row we get a row of zeroes. It follows that the rank of A is at most one and so A cannot be invertible. Now suppose that $bd = 0$. If $b = 0$ and $d \neq 0$ then $a = 0$ and again we have a row of zeroes. Similarly if $d = 0$ and $b \neq 0$. Finally if $b = d = 0$ then the second row is a multiple of the first row and Gaussian elimination again produces a row of zeroes.

4. (i) False. Let

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A has shape 2×1. A left inverse has shape 1×2. Let

$$B = \begin{pmatrix} a & b \end{pmatrix}.$$

We want $BA = I_1 = (1)$. This is equivalent to the linear equation $a + b = 1$. So

$$B_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

are two different left inverses of A.
(ii) Suppose that B is a left inverse of A and that v is a solution of the linear equation $Ax = b$. Then

$$b = Av$$

Multiplying both sides by B

$$\begin{aligned} Bb &= B(Av) \\ &= (BA)v \\ &= v. \end{aligned}$$

So if there is a solution it must be the vector Bb.
(iii) Since every equation has at most one solution, there must be more equations than variables, that is $m \geq n$.
5. (i) Let A be a $m \times n$ matrix. We say that a $n \times m$ matrix B is a **right inverse** if $AB = I_m$.
(ii) Every equation $Ax = b$ has at least one solution. Indeed, given b let $v = Bb$. Then

$$\begin{aligned} Av &= A(Bb) \\ &= (AB)b \\ &= b. \end{aligned}$$

So the vector $v = Bb$ is always a solution to the equation $Ax = b$.
(iii) Since every equation has at least one solution, there must be more variables than equations, that is $m \leq n$.
6. (a) Suppose not, suppose that M is an inverse of N. Pick $k > 0$ minimal such that $N^k = 0$. Then

$$I_n = NM$$

Multiplying both sides by N^{k-1} we get

$$\begin{aligned} 0 \neq N^{k-1} &= N^{k-1}(NM) \\ &= N^k M \\ &= 0, \end{aligned}$$

a contradiction. Therefore N is not invertible.

(b) Let $A = I_n - N$ and $B = I_n + N + N^2 + \cdots + N^{k-1}$. Then

$$\begin{aligned} AB &= (I_n - N)(I_n + N + N^2 + \cdots + N^{k-1}) \\ &= I_n + N + N^2 + \cdots + N^{k-1} - N(I_n + N + N^2 + \cdots + N^{k-1}) \\ &= I_n + N + N^2 + \cdots + N^{k-1} - N - N^2 - N^3 - \cdots N^{k-1} - 0 \\ &= I_n. \end{aligned}$$

Similarly $BA = I_n$. Hence B is the inverse of A.
(c) As A and N commute, we have

$$\begin{aligned} (AN)^k &= A^k N^k \\ &= 0. \end{aligned}$$

Therefore $-AN$ is nilpotent. But then $I_n + AN$ is invertible by (b).
7. (a) The elementary permutation matrices and $E_i(-1)$, for any i.
(b) Let P be a permutation matrix.

Claim 0.1. *There are elementary permutation matrices $P_1, P_2, \dots, P_k$ such that $P_k P_{k-1} \cdots P_1 P = I_n$.*

Proof. If $P = I_n$ then we may take $k = 0$ (that is there is nothing to prove). Otherwise let row i be the first row such that the (i, i) entry is not one. As $P \neq I_n$, $i < n$. The proof proceeds by descending induction on i.
By assumption row j contains a 1 in the the ith column, for some $j > i$. The matrix $Q = P_{i,j} P$ then has a 1 in the (i, i) entry. By induction we may find elementary permutation matrices P_2, P_3, $\dots$, P_k such that $P_k P_{k-1} \cdots P_2 Q = I_n$. But then $P_k P_{k-1} \cdots P_1 P = I_n$, where $P_1 = P_{i,j}$. ∎

Using the claim, we may find $P_1, P_2, \dots, P_k$ such that $P_k P_{k-1} \cdots P_1 P = I_n$. But then $P = P_1 P_2 \cdots P_k$.
To obtain a permutation matrix P, take the identity matrix and permute its rows. There are n possible places to put the first row. Having decided where to put the first row, there are $n - 1$ possible places to put the second row (the only thing we cannot do is put the second row in the same row as we decided to place the first row). Continuing in this way, we see that there are

$$n(n-1)(n-2)\cdots 1 = n!,$$

different permutation matrices.
(c) We want to show that P^t is the inverse Q of P. First suppose that $P = P_{i,j}$ is an elementary permutation matrix. Then

$$P^t = P_{i,j} \qquad \text{and} \qquad Q = P_{i,j},$$

and the result is clear. Otherwise we may find elementary permutation matrices $P_1, P_2, \dots, P_k$ such that $P = P_1 P_2 \dots P_k$. But then

$$\begin{aligned} Q &= P_k P_{k-1} P_{k-2} \cdots P_1 \\ &= P_k^t P_{k-1}^t \cdots P_1^t \\ &= (P_1 P_2 \cdots P_k)^t = P^t. \end{aligned}$$