MODEL ANSWERS TO THE SECOND HOMEWORK

1. In matrix notation, we may represent this system of linear equations as $Ax = b$, where

$$A = \begin{pmatrix} 3 & -2 & 4 & -1 \\ -2 & 3 & -2 & -2 \\ 5 & 0 & 8 & -7 \end{pmatrix} \qquad b = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \text{and} \qquad x = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}.$$

We try to solve this system. The augmented matrix is

$$\begin{pmatrix} 3 & -2 & 4 & -1 & | & a \\ -2 & 3 & -2 & -2 & | & b \\ 5 & 0 & 8 & -7 & | & c \end{pmatrix}.$$

We could try to solve this system using Gaussian elimination. However it is clear that this will be a mess. This suggests that we should look for another way. Now we have three equations in four unknowns. In general we always expect a solution. In fact the only thing that could go wrong is if the rows of the coefficient matrix are dependent, that is one row is a combination of the others. Simply using the method of guess and check, we see that three times the first row plus twice the second row is equal to the third row.

So let us take the first row, multiply it by -3 and add it to the third row. We get

$$\begin{pmatrix} 3 & -2 & 4 & -1 & | & a \\ -2 & 3 & -2 & -2 & | & b \\ -4 & 6 & -4 & -4 & | & c-3a \end{pmatrix}.$$

Now take the second row and multiply by -2 and add it the third row to get

$$\begin{pmatrix} 3 & -2 & 4 & -1 & | & a \\ -2 & 3 & -2 & -2 & | & b \\ 0 & 0 & 0 & 0 & | & c-3a-2b \end{pmatrix}.$$

If we turn the last row into an equation, we get

$$0 \cdot X + 0 \cdot Y + 0 \cdot Z + 0 \cdot W = c - 3a - 2b.$$

So at the very least we must have $c - 3a - 2b = 0$, that is $c = 3a + 2b$. On the other hand, if we multiply the first row by 2 and add it to the second row we get a matrix in echelon form, but with the columns in the wrong order (there will be a zero in $(2, 4)$ entry). It follows that we can solve this system of linear equations for a, b and c, using back substitution, provided $c = 3a + 2b$.

Here is another way to proceed, which again gets around the unpalatable arithmetic of a direct approach. Let us reorder the variables, which is equivalent to reordering the columns of A,

$$A = \begin{pmatrix} -1 & -2 & 4 & 3 \\ -2 & 3 & -2 & -2 \\ -7 & 0 & 8 & 5 \end{pmatrix} \qquad b = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \text{and} \qquad x = \begin{pmatrix} W \\ Y \\ Z \\ X \end{pmatrix}.$$

Here we switched the first and fourth columns, so we switch X and W. This gives us the augmented matrix

$$\begin{pmatrix} -1 & -2 & 4 & 3 & | & a \\ -2 & 3 & -2 & -2 & | & b \\ -7 & 0 & 8 & 5 & | & c \end{pmatrix}.$$

Now apply Gaussian elimination. We multiply the first row by -1.

$$\begin{pmatrix} 1 & 2 & -4 & -3 & | & -a \\ -2 & 3 & -2 & -2 & | & b \\ -7 & 0 & 8 & 5 & | & c \end{pmatrix}.$$

Now multiply the first row by 2 and 7 and add it to the second and third rows,

$$\begin{pmatrix} 1 & 2 & -4 & -3 & | & -a \\ 0 & 7 & -10 & -8 & | & b-2a \\ 0 & 14 & 20 & -16 & | & c-7a \end{pmatrix}.$$

Now take the second row and multiply it by -2 to get

$$\begin{pmatrix} 1 & 2 & -4 & -3 & | & -a \\ 0 & 7 & -10 & -8 & | & b-2a \\ 0 & 0 & 0 & 0 & | & c-3a-2b \end{pmatrix},$$

and finish the argument as before.

2. We want three planes that form the shape of a "toblerone". The three planes $x = 0$, $y = 0$ and $x + y = 1$ have the right shape. The resulting matrix is

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 1 & 1 & 0 & | & 1 \end{pmatrix}.$$

The first two equations have the solution set

$$\{\, (0, 0, z) \mid z \in \mathbb{R} \,\},$$

the first and third have the solution set

$$\{ (0, 1, z) \mid z \in \mathbb{R} \},$$

and the second and third have the solution set

$$\{ (1, 0, z) \mid z \in \mathbb{R} \}.$$

It is clear that the intersection of these sets is empty, so that the system of three linear equations has no solutions.

3. A general 2×2 matrix has the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The idea is to make particular choices for the matrix B and see what conditions they impose on a, b, c and d. Let

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which is an elementary matrix. By direct calculation,

$$AB = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$

The second equality follows from the fact that multiplying by B has the effect of adding the second row to the first row. If $AB = BA$ then equating the $(1, 1)$ entries we get

$$a = a + c.$$

This implies $c = 0$. Equating the $(1, 2)$ entries gives

$$a + b = b + d.$$

This implies $a = d$. Thus

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Now take

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the other obvious elementary matrix to try. In this case

$$AB = \begin{pmatrix} a+b & b \\ a & a \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} a & b \\ a & a+b \end{pmatrix}.$$

Equating the $(1, 1)$ entries gives

$$a + b = a.$$

This implies $b = 0$. It follows that if A commutes with every matrix then A must have the form

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

4. Let A be an $n \times n$ matrix. If A commutes with every other $n \times n$ matrix B then A is a scalar multiple of the identity, that is $A = \lambda I_n$. To see why this is true, pick any two indices $1 \leq i \leq n$ and $1 \leq j \leq n$, $i \neq j$ and let $B = E_{ij}(1)$. Then the (i, i) entry of AB is

$$\sum_{i,l} a_{il} b_{li} = a_{ii} + a_{ij},$$

and the (i, i) entry of BA is

$$\sum_{i,l} b_{il} a_{li} = a_{ii}.$$

Equating these two entries gives

$$a_{ii} + a_{ij} = a_{ii}.$$

This implies $a_{ij} = 0$. Since i and j are arbitrary (subject to not being equal) this implies that every off-diagonal entry is zero, that is A is a diagonal matrix.
Now let us compare the (j, i) entries. The (j, i) entry of AB is

$$\sum_{i,l} a_{jl} b_{li} = a_{jj} b_{ji} = a_{jj},$$

and the (j, i) entry of BA is

$$\sum_{i,l} b_{jl} a_{li} = b_{ji} a_{ii} = a_{ii}.$$

Equating these two entries gives

$$a_{ii} = a_{jj}.$$

But then the diagonal entries are all equal to the same value λ. In this case $A = \lambda I_n$.
5. By symmetry we may suppose that the first system has infinitely many solutions. By assumption the second system has at least one solution $v_2 \in \mathbb{R}^n$. Let $v_1 \in \mathbb{R}^n$ be any solution to the first equation. Then

$$\begin{aligned} A(v_1 + v_2) &= Av_1 + Av_2 \\ &= b_1 + b_2. \end{aligned}$$

Thus the sum $v_1 + v_2$ is a solution to the equation $Ax = b_1 + b_2$. Fixing v_2 and varying v_1 we get infinitely many solutions to the equation $Ax = b_1 + b_2$.

6. (a) The matrix PA is obtained from the matrix A by swapping the second and third rows.

(b) The matrix QA is obtained from the matrix A by adding the second and third rows to A and adding the third row to the second row.