MODEL ANSWERS TO THE FIRST HOMEWORK

1. Every cubic polynomial with real coefficients has the form $f(t) = a + bt + ct^2 + dt^3$, where $(a, b, c, d) \in \mathbb{R}^3$ are constants to be determined. Note that $f'(t) = b + 2ct + 3dt^2$. We have

$$\begin{aligned} 12 &= f(2) = a + 2b + 4c + 8d \\ 40 &= f(3) = a + 3b + 9c + 27d \\ 5 &= f'(1) = b + 2c + 3d \\ 17 &= f'(2) = b + 4c + 12d. \end{aligned}$$

Thus we get the system of linear equations $Av = b$, where

$$A = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 12 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 12 \\ 40 \\ 5 \\ 17 \end{pmatrix}.$$

We solve this system using Gaussian elimination. We start with the augmented matrix,

$$\begin{pmatrix} 1 & 2 & 4 & 8 & | & 12 \\ 1 & 3 & 9 & 27 & | & 40 \\ 0 & 1 & 2 & 3 & | & 5 \\ 0 & 1 & 4 & 12 & | & 17 \end{pmatrix}.$$

Multiplying the first row by -1 and adding to the second row gives

$$\begin{pmatrix} 1 & 2 & 4 & 8 & | & 12 \\ 0 & 1 & 5 & 19 & | & 28 \\ 0 & 1 & 2 & 3 & | & 5 \\ 0 & 1 & 4 & 12 & | & 17 \end{pmatrix}.$$

Multiplying the second row by -1 and adding to the third and fourth rows gives

$$\begin{pmatrix} 1 & 2 & 4 & 8 & | & 12 \\ 0 & 1 & 5 & 19 & | & 28 \\ 0 & 0 & -3 & -16 & | & -23 \\ 0 & 0 & -1 & -7 & | & -11 \end{pmatrix}.$$

Multiplying the third row by $-1/3$ gives

$$\begin{pmatrix} 1 & 2 & 4 & 8 & | & 12 \\ 0 & 1 & 5 & 19 & | & 28 \\ 0 & 0 & 1 & 16/3 & | & 23/3 \\ 0 & 0 & -1 & -7 & | & -11 \end{pmatrix}.$$

Adding the third row to the fourth row gives

$$\begin{pmatrix} 1 & 2 & 4 & 8 & | & 12 \\ 0 & 1 & 5 & 19 & | & 28 \\ 0 & 0 & 1 & 16/3 & | & 23/3 \\ 0 & 0 & 0 & -5/3 & | & -10/3 \end{pmatrix}.$$

Multiplying the fourth row by $-3/5$ gives

$$\begin{pmatrix} 1 & 2 & 4 & 8 & | & 12 \\ 0 & 1 & 5 & 19 & | & 28 \\ 0 & 0 & 1 & 16/3 & | & 23/3 \\ 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}.$$

This completes the elimination. Now we use back substitution to solve for (a, b, c, d). The last equation implies that $d = 2$. Using this value of d, the third equation reads,

$$c + 32/3 = 23/3,$$

so that $c = -3$. Using these two values of c and d in the second equation gives,

$$b - 15 + 38 = 28,$$

so that $b = 5$. The first equation then reads

$$a + 10 - 12 + 16 = 12,$$

so that $a = -2$. Thus $f(t) = 2t^3 - 3t^2 + 5t - 2$. As a check, $f'(t) = 6t^2 - 6t + 5$ and so $f'(2) = 24 - 12 + 5 = 17$, as required.

Note that if we switch the order of (a, b, c, d), which is equivalent to reversing the columns of A then trying to apply Gaussian elimination is much more cumbersome.

2. We apply Gaussian elimination. The augmented matrix is

$$\begin{pmatrix} 2 & -1 & -3 & | & 1 \\ 4 & -3 & -1 & | & -2 \\ -6 & 2 & 14 & | & 2t - 1 \end{pmatrix}.$$

Let us cheat a little and not bother to create a 1 in the first entry (this simplifies the algebra but does not change the theory). Take the first row and multiply it by -2 and 3 and add it to the second and third rows to get

$$\begin{pmatrix} 2 & -1 & -3 & | & 1 \\ 0 & -1 & 5 & | & -4 \\ 0 & -1 & 5 & | & 2t+2 \end{pmatrix}.$$

Now multiply the second row by -1 and add it to the third row to get

$$\begin{pmatrix} 2 & -1 & -3 & | & 1 \\ 0 & -1 & 5 & | & 0 \\ 0 & 0 & 0 & | & 2t+6 \end{pmatrix}.$$

Since the first augmented matrix and the last augmented matrix are row equivalent, it follows that they have the same solutions. Since the last row is a row of zeroes for the coefficient matrix, the only way that this system of linear equations has a solution is if $2t + 6 = 0$ as well. But then $t = -3$. On the other hand if $t = -3$ then it is easily to see by back subsitution that there are infinitely many solutions.

3. (i) False. Suppose that $A = (0, 0)$ and $b = (1)$. Then $m = 1 < 2 = n$, but the equation

$$0x + 0y = 1,$$

does not have any solutions.

(ii) True. For example, the zero vector of type $n \times 1$ is clearly a solution.

(iii) True. Apply Gaussian elimination to the augmented matrix $C = (A|b)$. At the end, we arrive at $C' = (A'|b')$ where A' is in echelon form. The two augmented matrices C and C' are row equivalent, so they have the same solutions. By assumption there is a solution, so that if the ith row of A' is a row of zeroes, then the ith row of C' is a row of zeroes as well. The corresponding equation imposes no conditions and so we can ignore these rows. For each of the other rows, using the pivot one can express the corresponding variable in terms of the other variables. Since the number of pivots is at most the number of rows, it follows that some variables are free. When we apply back substitution we are free to choose any values for these variables. We can assign to each free variable any real number and since there are infinitely many real numbers there are infinitely many solutions.