

4. MATRICES AS ELEMENTARY TRANSFORMATIONS

Definition 4.1. Let m be a positive integer. Let i and j be any two integers $1 \leq i, j \leq m$ and let λ be any real number.

If $\lambda \neq 0$ then let $E_i(\lambda)$ be the $m \times m$ square matrix with (a, b) entry

$$\begin{cases} \lambda & \text{if } a = b = i \\ 1 & \text{if } a = b \neq i \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $i \neq j$. Let $E_{i,j}(\lambda)$ be the $m \times m$ square matrix with (a, b) entry

$$\begin{cases} \lambda & \text{if } (a, b) = (j, i) \\ 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_{i,j}$ be the $m \times m$ square matrix with (a, b) entry

$$\begin{cases} 1 & \text{if either } (a, b) = (i, j) \text{ or } (a, b) = (j, i) \\ 1 & \text{if } a = b \neq i, j \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the matrix $P_{i,j}$ is obtained from the matrix I_n by switching the i th and j th rows, the matrix $E_i(\lambda)$ is obtained from the matrix I_n by multiplying the i th row by λ and the matrix $E_{i,j}(\lambda)$ is obtained from the matrix I_n by changing the (j, i) entry to λ .

Lemma 4.2. Let m and n be two positive integers. Let i and j be any two integers $1 \leq i, j \leq m$ and let λ be any real number.

Let A be an $m \times n$ matrix. Then the matrix

- (1) $E_i(\lambda)A$ is obtained from the matrix A by multiplying the i th row by λ .
- (2) $E_{i,j}(\lambda)A$ is obtained from the matrix A by multiplying the i th row of A by λ and adding it the j th row.
- (3) $P_{i,j}A$ is obtained from the matrix A by switching the i th and the j th rows.

Proof. Easy calculation left to any student taking 18.700. □

In other words, the elementary row operations are represented by multiplying by the corresponding elementary matrix.

Definition 4.3. Let m be a positive integer. Let L be an $m \times m$ matrix. We say that L is **lower triangular** if $a_{ij} = 0$ if $i < j$.

Theorem 4.4. Let m and n be any positive integers and let A be a $m \times n$ matrix.

Then we may write

$$A = PLU,$$

where P is a $m \times m$ permutation matrix (a product of elementary permutation matrices) L is a lower triangular $m \times m$ matrix and U is a $m \times n$ matrix in echelon form.

We need some easy:

Lemma 4.5. Let n be a positive integer and let A_1, A_2, \dots, A_k be a sequence of invertible matrices of type $n \times n$ with inverses B_1, B_2, \dots, B_k .

Then the product matrix $A = A_1 A_2 A_3 \cdots A_k$ is invertible with inverse $B = B_k B_{k-1} \cdots B_1$.

Proof. By an obvious induction, it suffices to prove the case $k = 2$. We compute

$$\begin{aligned} BA &= (B_2 B_1)(A_1 A_2) \\ &= (B_2(B_1 A_1))A_2 \\ &= (B_2 I_n)A_2 \\ &= B_2 A_2 = I_n. \end{aligned}$$

Similarly one can check $AB = I_n$. □

Lemma 4.6. The product of lower triangular matrices is lower triangular.

Proof. By an obvious induction it suffices to prove that if $A = (a_{ij})$ and $B = (b_{ij})$ are two lower triangular matrices of shape $n \times n$ then so is the product AB . Suppose $1 \leq i < j \leq n$. Now if $l < j$ then $b_{lj} = 0$ as B is lower triangular. If $l > i$ then $a_{il} = 0$ as A is lower triangular. The (i, j) entry of AB is

$$\begin{aligned} \sum_l a_{il} b_{lj} &= \sum_{l \leq i} a_{il} b_{lj} + \sum_{i < l < j} b_{il} b_{lj} + \sum_{l \geq j} a_{il} b_{lj} \\ &= \sum_{l \leq i} a_{il} \cdot 0 + \sum_{i < l < j} 0 \cdot 0 + \sum_{l \geq j} 0 \cdot b_{lj} \\ &= 0. \end{aligned}$$

But then AB is lower triangular. □

Proof of (4.4). Apply Gaussian elimination to A . Suppose that U is the end result of Gaussian elimination. Then U is a $m \times n$ matrix

in echelon form. Let E_1, E_2, \dots, E_s be the elementary matrices corresponding to the steps of Gaussian elimination and let E' be the product,

$$E' = E_s E_{s-1} \cdots E_2 E_1.$$

Then

$$E'A = U.$$

The first thing to observe is that one can change the order of some of the steps of the Gaussian elimination. Some of the matrices E_i are elementary permutation matrices corresponding to swapping two rows. In fact, one can perform all of these operations first (in the same order) and then apply elimination. Let Q be the product of all of these matrices and let E be the product of the rest. Then

$$(EQ)A = U.$$

Now let P be the product of all the permutation matrices but now in the opposite order and let L be the product of the inverse elementary transformations, in the reverse order. By (4.5) PL is the inverse matrix of EQ (note that $P_{i,j}$ is its own inverse). We have

$$\begin{aligned} A &= I_m A \\ &= ((PL)(EQ))A \\ &= PL((EQ)A) \\ &= (PL)U \\ &= PLU. \end{aligned}$$

Note that the elementary matrices $E_i(\lambda)$ and $E_{i,j}(\lambda)$ corresponding to the elementary row operations that appear in Gaussian elimination are all lower triangular. On the other hand, since one can undo any elementary row operation by an elementary row operation of the same type, these matrices are invertible and their inverses are of the same type. Since L is a product of such matrices, (4.6) implies that L is lower triangular. \square

(4.4) can be turned into a very efficient method to solve linear equations.

For example suppose that we start with the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 2 & -2 & 6 & 2 \\ 1 & -1 & 5 & -1 \end{pmatrix}.$$

Now if we apply Gaussian elimination straight away then we get a row of zeroes and we need to swap the second and third rows. In order to

make sure that all permutations go at the front, the first thing we do is swap the second and third rows,

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 1 & -1 & 5 & -1 \\ 2 & -2 & 6 & 2 \end{pmatrix}.$$

Okay, now we apply Gaussian elimination. We multiply the first row by -1 and -2 add it to the second and third rows,

$$\begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we multiply the second row by $1/2$ to get

$$U = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This completes the Gaussian elimination. The corresponding elementary matrices, written in the correct order are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The inverse of this product is then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

that is the inverse of the individual elementary matrices, taken in the opposite order. Therefore

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Putting all of this together, we get

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 2 & -2 & 6 & 2 \\ 1 & -1 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = PLU.$$

Theorem 4.7. *Let A and B be two $m \times n$ matrices.*

If A and B are row equivalent then there are elementary matrices E_1, E_2, \dots, E_s such that

$$B = E_s E_{s-1} \cdots E_2 E_1 A.$$

Proof. This statement is implicit in the proof of (4.4), cf. the book. \square

Lemma 4.8. *Let A and B be two matrices in row echelon form.*

If A and B are row equivalent then they have same number of pivots.

Proof. We proceed by induction on m . If every row of A and B contains a pivot there is nothing to prove. In particular we may assume that $m > 0$.

By symmetry we may assume that A has a row of zeroes. Since A and B are row equivalent, for every $b \in \mathbb{R}^m$ there is another vector $c \in \mathbb{R}^m$ such that the equations

$$Ax = b \quad \text{and} \quad Bx = c,$$

have the same solutions.

Suppose that the i th row of A is a row of zeroes. Let b be the $m \times 1$ vector whose i th row is 1 and whose other entries are zero. Then the system $Ax = b$ has no solutions. Therefore the system $Bx = c$ has no solutions, for some unknown vector c . It follows that some row of B is a row of zeroes, else we could solve the system $Bx = c$ using back substitution.

Since A and B are in echelon form the last row of A and the last row of B are both rows of zeroes. Let A' be the submatrix of A obtained by deleting the last row of A and let B' be the submatrix of B obtained by deleting the last row of B . Then A' and B' are row equivalent. Since they have $m - 1$ rows, by induction A' and B' have the same number of pivots. But A' and A have the same number of pivots and the same is true of B and B' . \square

Definition 4.9. *Let A be a matrix.*

*The **rank** of A , denoted $\text{rk}(A)$ is the number of pivots for any matrix U in echelon form, which is row equivalent to A .*

Note that (4.8) implies that the rank is well defined:

Lemma 4.10. *Let A be a matrix and let U_1 and U_2 be two matrices in row echelon form which are row equivalent to A .*

Then U_1 and U_2 have the same number of pivots. In particular the rank is well defined.

Proof. Since U_1 and U_2 are row equivalent to A they are row equivalent to each other. Now apply (4.7). \square