18. Canonical forms III

Definition 18.1. *Let $\phi\colon V \longrightarrow W$ be a linear map between two real inner product spaces. We say that ϕ is an* ***isometry*** *if ϕ respects the inner products, that is*

$$\langle u, v\rangle = \langle \phi(u), \phi(v)\rangle,$$

for every u and $v \in V$.

Note that rotation and flips are isometries of $\mathbb{R}^2$. Note also that ϕ is an isometry if and only if ϕ respects the norms, that is

$$\|v\| = \|\phi(v)\|.$$

This is because one can recover the inner product from the norm.

Lemma 18.2. *Let $\phi\colon V \longrightarrow W$ be an isometry.*
Then ϕ is injective.

Proof. It suffices to prove that the kernel is trivial.
Suppose that $v \in \operatorname{Ker}\phi$. Then

$$\langle v, v\rangle = \langle \phi(v), \phi(v)\rangle = 0.$$

But then $v = 0$ and $\operatorname{Ker}\phi = \{0\}$. □

Theorem 18.3. *Let V be a real inner product space of dimension n.*
Then V is isometric to $\mathbb{R}^n$ (with the standard inner product).

Proof. Pick any linear isomorphism $f\colon V \longrightarrow \mathbb{R}^n$. Define an inner product on $\mathbb{R}^n$ using the inner product on V,

$$\langle u, v\rangle = \langle f^{-1}(u), f^{-1}(v)\rangle.$$

It is easy to check that this does define an inner product on $\mathbb{R}^n$. So it is enough to show that any inner product on $\mathbb{R}^n$ is equivalent to the standard inner product.

Pick any basis $v_1, v_2, \dots, v_n$ of $\mathbb{R}^n$. Applying Gram-Schmidt we may assume that this basis is orthonormal. But then

$$\langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and this inner product is equivalent to the standard one. In fact the isometry is given by the linear map given by the orthonormal basis, $v_1, v_2, \dots, v_n$. □

Lemma 18.4. *Let $A \in M_{n,n}(\mathbb{R})$. Let u and $v \in \mathbb{R}^n$. Then*

$$\langle u, Av\rangle = \langle A^t u, v\rangle.$$

Proof. This is easy:

$$\begin{aligned}\langle A^t u, v\rangle &= (A^t u)^t v \\ &= u^t A v \\ &= \langle u, Av\rangle. \qquad \square\end{aligned}$$

Definition 18.5. *If A is invertible and $A^{-1} = A^t$ then we say that A is* ***orthogonal****.*

Proposition 18.6. *Let $A \in M_{n,n}(\mathbb{R})$. Let $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the corresponding linear function.*

Then ϕ is an isometry of $\mathbb{R}^n$ (with the standard inner product) iff A is orthogonal.

Proof. Suppose that u and v are in $\mathbb{R}^n$. Suppose that ϕ is an isometry. By assumption

$$\begin{aligned}\langle u, v\rangle &= \langle \phi(u), \phi(v)\rangle \\ &= \langle Au, Av\rangle \\ &= \langle u, A^t A v\rangle.\end{aligned}$$

Therefore

$$\langle u, v - A^t A v\rangle = \langle u, v\rangle - \langle u, A^t A v\rangle.$$

As the inner product is non-degenerate, and u is arbitrary, it follows that $v - A^t A v = 0$, that is $v = A^t A v$. As v is aribtrary, the $A^t A = I_n$. But then A is invertible and the inverse is A^t. Therefore A is orthogonal.

Now suppose that A is orthogonal. Then

$$\begin{aligned}\langle \phi(u), \phi(v)\rangle &= \langle Au, Av\rangle \\ &= \langle u, A^t A v\rangle \\ &= \langle u, v\rangle.\end{aligned}$$

Thus ϕ is an isometry. $\square$

Note that it is very useful to know that A is orthogonal. For example,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

represents rotation through an angle of θ. The transpose is

$$R^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and we can see this is the inverse as this represents rotation through $-\theta$. On the other hand,

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

represents a flip about the line $y = x$. The tranpose is the same matrix and the inverse is also the same matrix. Note that the determinant of an orthogonal matrix is ± 1, since

$$\begin{aligned} 1 &= \det I_n \\ &= \det AA^t \\ &= (\det A)(\det A^t) \\ &= (\det A)^2. \end{aligned}$$

Rotations have determinant one; the rest have determinant -1.