10. Rank-nullity

Definition 10.1. *Let $A \in M_{m,n}(F)$. The **row space** of A is the span of the rows. The **column space** of A is the span of the columns. The **nullity** $\nu(A)$ of A is the dimension of the kernel.*

The dimension of the row space is the number of independent conditions imposed by the equations determined by the rows of Ax. The column space is the image of ϕ, that is the set of vectors b such that the equation $Ax = b$ has a solution.

Theorem 10.2 (Rank-Nullity)**.** *Let $A \in M_{m,n}(F)$. Let r be the rank of A and let ν be the nullity.*

Then $\nu + r = n$.

We have already seen many examples of the rank-nullity formula, when we looked at the function associated to a matrix. For example, rank-nullity says that a single (non-zero) equation in 3 variables defines a plane in $\mathbb{R}^3$. Two such equations define a line, unless one equation is a multiple of the other. A hyperplane in F^d (that is the linear subspace defined by a single non-zero equation) has dimension $d-1$. And so on.

We need a couple of preparatory:

Lemma 10.3. *Let $U \in M_{m,n}(F)$ be a matrix in echelon form.*

Then the non-zero rows $v_1, v_2, \dots, v_r \in F^n$ of U are independent.

Proof. Suppose not, suppose that $v_1, v_2, \dots, v_r$ are dependent. We will derive a contradiction. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_r$ are scalars, not all zero such that

$$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r.$$

Let i be the smallest index such that $\lambda_i \neq 0$. Then

$$0 = \lambda_i v_i + \lambda_{i+1} v_{i+1} + \cdots + \lambda_r v_r.$$

Suppose that there is a pivot in the (i, j) entry. Then the jth entry of the RHS is equal to λ_i, which is not zero, a contradiction. Thus the vectors $v_1, v_2, \dots, v_r$ are independent. □

Lemma 10.4. *Let $U \in M_{m,n}(F)$ be a matrix in reduced row echelon form of rank r.*

The dimension of the row space, the column space and the rank are all equal to r. The nullity ν is equal to $n-r$. In particular (10.2) *holds for any matrix in reduced row echelon form.*

Proof. The rank is the number of pivots r.

The non-zero rows of U obviously span the row space and the number of non-zero rows is equal to the number of pivots. By (10.3) they are also independent, and so the non-zero rows are a basis for the row space. Therefore the dimension of the row space is r.

Let $w_1, w_2, \dots, w_r \in F^m$ be the columns of U which contain a pivot. If we discard the rows which don't contain a pivot, then we don't change the dimension of the column space, since we are only throwing away zeroes. The vectors $w_1, w_2, \dots, w_r$ are then the standard basis of F^r. It follows that the column space is the whole of $F^r = F^n$ (remember we threw away a lot of rows) and so the column space has dimension r as well.

Finally we compute the nullity ν. To solve the homogeneous equation $Ux = 0$ we apply back substitution. The variables which correspond to the columns which don't contain a pivot are the free variables. There are $t = n - r$ such variables. Applying back substitution we generate vectors $u_1, u_2, \dots, u_t$ such that the general solution of the homogeneous is

$$\mu_1 u_1 + \mu_2 u_2 + \cdots + \mu_t u_t.$$

Hence the vectors $u_1, u_2, \dots, u_t$ span the kernel. Let $U' \in M_{t,n}(F)$ be the matrix whose rows are the vectors $u_1, u_2, \dots, u_t$. The matrix U' is not in echelon form, but it is close. The last non-zero entry in every row is one and the entries above this row are all zero. If we reverse the order of the rows and the columns this clearly won't change the rank of the row space of U'. On the other hand the resulting matrix U'' is in echelon form. By (10.3) it follows that the vectors $u_1, u_2, \dots, u_t$ are independent. Thus the vectors $u_1, u_2, \dots, u_t$ are a basis for the kernel and so the nullity is $n - r$. $\square$

It might be helpful to consider an example of how the proof of the last statement works. Consider the following matrix

$$U = \begin{pmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider determing the nullity. Suppose that the variables are x_1, x_2, x_3, x_4 and x_5. The rank of this matrix is two and there are three free variables, x_2, x_4 and x_5, corresponding to the columns which don't contain pivots. Solving by back substitution, we first express x_3 in terms of x_4 and x_5,

$$x_3 = x_4 - x_5.$$

Now solve for x_1 in terms of x_2, x_4 and x_5.

$$x_1 = x_2 + 2x_4 - 3x_3.$$

The general solution to the homogeneous is of the form

$$(x_2 + 2x_4 - 3x_3, x_2, x_4 - x_5, x_4, x_5).$$

If we separate out x_2, x_4 and x_5, we can write this in the form

$$x_2(1, 1, 0, 0, 0) + x_4(2, 0, 1, 1, 0) + x_5(0, 0, -1, 0, 1) = r_1 u_1 + r_2 u_2 + r_3 u_3,$$

where $r_1 = x_2$, $r_2 = x_4$, and $r_3 = x_5$ and

$$u_1 = (1, 1, 0, 0, 0) \qquad u_2 = (2, 0, 1, 1, 0) \qquad u_3 = (0, 0, -1, 0, 1).$$

The vectors u_1, u_2 and u_3 visibly span the kernel. We want to show that they are independent. Put them into a matrix

$$U' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

We would like to appeal to (10.3). The problem is that this matrix does not quite have the right shape (it is upside down echelon form, if you will). We can either say that it is easy to modify the proof to this case, or, even better, if we simply reverse the order of the rows, we get

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and then the columns, we get

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

which is a matrix in echelon form. On the other hand these operations don't change the rank.

Lemma 10.5. *Let A and $B \in M_{m,n}(F)$.*

If A and B are row equivalent then

1. $\operatorname{Ker} A = \operatorname{Ker} B$.
2. *the row space of A is equal to the row space of B.*
3. *the dimension of the column space of A is equal to the dimension of the column space of B.*

Proof. If A and B are row equivalent then so are the augmented matrices $(A \,|\, 0)$ and $(B \,|\, 0)$. But then the solution to the homogeneous equations $Ax = 0$ and $Bx = 0$ are the same. This is (1).

We now prove (2). By induction we may assume that we can get from A to B by one elementary row operation. Permuting two rows obviously leaves the row space unchanged. Suppose that we multiply the ith row by $\lambda \neq 0$. Let $v_1, v_2, \dots, v_m$ be the rows of A and let $w_1, w_2, \dots, w_m$ be the rows of B. Thus

$$w_j = \begin{cases} v_j & j \neq i \\ \lambda v_j & j = i. \end{cases}$$

Suppose that $v \in \operatorname{span}\{v_1, v_2, \dots, v_m\}$. Then there are scalars $r_1, r_2, \dots, r_m$ such that

$$v = r_1 v_1 + r_2 v_2 + \dots + r_m v_m.$$

For ease of notation assume $i = m$. Then

$$v = r_1 w_1 + r_2 w_2 + \dots + s_m v_m,$$

where $s_m = r_m/\lambda$. Thus $v \in \operatorname{span}\{w_1, w_2, \dots, w_m\}$. We have shown

$$\operatorname{span}\{v_1, v_2, \dots, v_m\} \subset \operatorname{span}\{w_1, w_2, \dots, w_m\}.$$

The reverse inclusion is just as easy and so the row spaces are the same in this case. Finally suppose that we take a multiple of one row and add it to another. Let us suppose that we take the $(m-1)$th row multiply it by λ and add it to the last row. Then

$$w_j = \begin{cases} v_j & j < m \\ v_m + \lambda w_{m-1} & j = m. \end{cases}$$

Suppose that $v \in \operatorname{span}\{v_1, v_2, \dots, v_m\}$. Then there are scalars $r_1, r_2, \dots, r_m$ such that

$$v = r_1 v_1 + r_2 v_2 + \dots + r_m v_m.$$

Then

$$v = r_1 v_1 + r_2 v_2 + \dots + r_{m-2} w_{m-2} + (r_{m-1} - \lambda r_m) w_{m-1} + r_m w_m.$$

Thus $v \in \operatorname{span}\{w_1, w_2, \dots, w_m\}$. We have shown

$$\operatorname{span}\{w_1, w_2, \dots, w_m\} \subset \operatorname{span}\{v_1, v_2, \dots, v_m\}.$$

The reverse inclusion is just as easy and so the row spaces are the same in this case. This proves (2).

We turn to (3). Suppose that columns $w_1, w_2, \dots, w_s$ of A are independent. Let A' be the matrix whose columns are these vectors. Let $u_1, u_2, \dots, u_s$ be the corresponding columns of B and let B' be the corresponding submatrix. As A and B are row equivalent, it follows that A' and B' are also row equivalent. Since the vectors $w_1, w_2, \dots, w_s$ are independent, the kernel of A' is the trivial subspace $\{0\}$. By what we have already proved, the same is true for the kernel of B'. But then the vectors $w_1, w_2, \dots, w_s$ are independent. It follows that the dimension of the column space of B is at least the dimension of the column space of A. By symmetry (3) holds. $\square$

Proof of (10.2). If we apply Gauss Jordan elimination then we see that A is row equivalent to a matrix in row reduced echelon form. By (10.5) this changes neither the rank nor the nullity (nor does it change the number of columns). But (10.4) states (10.2) holds for a matrix in reduced row echelon form. $\square$