1. Systems of linear equations

We are interested in the solutions to systems of linear equations. A linear equation is of the form

$$3x - 5y + 2z + w = 3.$$

The key thing is that we don't multiply the variables together nor do we raise powers, nor takes logs or introduce sine and cosines. A system of linear equations is of the form

$$\begin{aligned} 3x - 5y + 2z &= 3 \\ -2x + y + 5z &= -4. \end{aligned}$$

This is a system of two linear equations in three variables. The first equation is a system consisting of one linear equation in four variables.

In this class we will be more interested in the nature of the solutions rather than the exact solutions themselves. So let us try to form a picture of what to expect. Suppose we start with an easy case. A system of two linear equations in two variables:

$$\begin{aligned} x - 2y &= -1 \\ 2x + y &= 3. \end{aligned}$$

It is easy to check that this has the unique solution $x = y = 1$. To see that this is not the only qualitative behaviour, suppose we consider the system

$$\begin{aligned} 2x + y &= 3 \\ 2x + y &= 3. \end{aligned}$$

Since the second equation is precisely the same as the first equation, it is enough to find x and y satisfying the system

$$2x + y = 3.$$

In other words the solution set is infinite. Any point of the line given by the equation $2x + y = 3$ will do. The problem is that the second equation does not impose independent conditions. Note that we can disguise (to a certain extent) this problem by writing down the system

$$\begin{aligned} 2x + y &= 3 \\ 4x + 2y &= 6. \end{aligned}$$

All we did was take the second equation and multiply it by 2. But we are not fooled, we still realise that the second equation imposes no more conditions than the first. It is not independent of the first equation.

We can tweak this example to get a system

$$\begin{aligned} 2x + y &= 3 \\ 2x + y &= 2. \end{aligned}$$

Now this system has no solutions whatsoever. If $2x + y = 3$ then $2x + y \neq 2$. The key point is that these three examples exhaust the qualitative behaviour:

- One solution.
- Infinitely many solutions.
- No solutions.

In other words if there are at least two solutions then there are infinitely many solutions. In particular it turns out that the answer is never that there are two solutions. We say that a linear system is **inconsistent** if there are no solutions and otherwise we also say that the system is **consistent**.

It is instructive to think how this comes out geometrically. Suppose there are two variables x and y. Then we can represent the solutions to a system of linear equations in x and y as a set of points in the plane. Points in the plane are represented by a pair of points (x, y) and we will refer to these points as vectors. The set of all such points is called $\mathbb{R}^2$,

$$\mathbb{R}^2 = \{\, (x, y) \mid x, y \in \mathbb{R} \,\}.$$

Okay, so what are the possibilities for the solution set? Well suppose we have a system of one equation. The solution set is a line. Suppose we have a system of two equations. If the equations impose independent conditions, then we get a single point, in fact the intersection of the two lines represented by each equation. Are there other possible solution sets? Yes, we already saw we might get no solutions. Either take a system of two equations with no solutions (in fact parallel lines) or a system of three equations, where we get three different points when we intersect each pair of lines. If we have three equations can we still get a solution? Yes suppose all three lines are concurrent (pass through the same point). It is interesting to see how this works algebraically. Suppose we consider the system

$$\begin{aligned} 2x + y &= 3 \\ x - 2y &= -1 \\ 3x - y &= 2. \end{aligned}$$

Then the vector $(x, y) = (1, 1)$ still satisfies all three equations. Note that the third equation is nothing more than the sum of the first two equations. As soon as one realises this fact, it is clear that the third

equation fails to impose independent conditions, that is the third equation depends on the first two. Note one further possibility. The following represents a system of three linear equations in two variables, with a line of solutions:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \\ 3x + 3y &= 3. \end{aligned}$$

These equations are very far from being independent. In fact no matter how many equations there are, it is still possible (but more and more unlikely) that there are solutions.

Finally let me point out some unusual possibilities. If we have a system of no equations, then the solution set is the whole of $\mathbb{R}^2$. In fact

$$0x + 0y = 0,$$

represents a single equation whose solution set is $\mathbb{R}^2$. By the same token

$$0x + 0y = 1,$$

is a linear equation with no solutions.

In summary the solution set to a system of linear equations in two variables exhibits one of four different qualitative possibilities:

(1) No solutions.
(2) One solution.
(3) A line of solutions.
(4) The whole plane $\mathbb{R}^2$.

(Strictly speaking, so far we have only shown that these possibilities occur, we have not shown that these are the only possibilities.). In other words the case when there are infinitely many solutions can be be further refined into the last two cases.

Now let us delve deeper into characterising the possible behaviour of the solutions. Let us consider a system of three linear equations in three unknowns. Can we list all geometric possibilities? One possibility is we get a point. For example

$$\begin{aligned} x \qquad\quad &= 0 \\ y \quad &= 0 \\ z &= 0. \end{aligned}$$

The first equation cuts down the space of solutions to a plane, the plane

$$\{\, (0, y, z) \mid y, z \in \mathbb{R} \,\}.$$

The second equation represents another plane which intersects the first plane in a line

$$\{\,(0,0,z)\,|\,z \in \mathbb{R}\,\}.$$

The last equation defines a plane which intersects this line in the origin, $(0,0,0)$. Or the first two equations could give a line and the third equation might also contain the same line,

$$\begin{aligned} x \qquad\quad &= 0 \\ y \quad &= 0 \\ x + y \quad &= 0. \end{aligned}$$

The last equation is a sum of the first two equations. But we could just as well consider

$$\begin{aligned} x \qquad\quad &= 0 \\ y \quad &= 0 \\ 3x - 2y \quad &= 0. \end{aligned}$$

Again the problem is that the third equation is not independent from the first two equations. There are many other possibilities. Two planes could be parallel, in which case there are no solutions.

It is time for some general notation and definitions. A vector

$$(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n,$$

is a **solution to a linear equation**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

if

$$a_1r_1 + a_2r_2 + \cdots + a_nr_n = b.$$

A system of m linear equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \;+ \quad \vdots \quad + \ddots + \quad \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

A **solution to a system of linear equations** is a vector $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ which satisfies all m equations simultaneously. The **solution set** is the set of all solutions.

Okay let us now turn to the boring bit. Given a system of linear equations, how does one solve the system? Again, let us start with a

simple example,

$$\begin{aligned} x - 2y &= -1 \\ 2x + y &= 3. \end{aligned}$$

There are many ways to solve this system, but let me focus on a particular method which generalises well to larger systems. Use the first equation to eliminate x from the second equation. To do this, take the first equation, multiply it by -2 and add it to the second equation (we prefer to think of multiplying by a negative number and adding rather than subtracting). We get a new system of equations,

$$\begin{aligned} x - 2y &= -1 \\ 5y &= 5. \end{aligned}$$

Since the second equation does not involve x at all, it is straightforward to use the second equation to get $y = 1$. Now use that value and substitute it into the first equation to determine x,

$$x - 2 = -1,$$

so that $x = 1$. This method is called **Gaussian elimination**. The last step, where we go from the bottom to the top and recursively solve for the variables is called **backwards substitution**.

Let us do the same thing with a system of three equations in three variables:

$$\begin{aligned} x - 2y - z &= 4 \\ 2x - 3y + z &= 10 \\ -x + 5y + 11z &= 3. \end{aligned}$$

We start with the first equation and use it to eliminate the appearance of x from the second equation. To do this, take the first equation, multiply it by -2 and add it to the second equation.

$$\begin{aligned} x - 2y - z &= 4 \\ y + 3z &= 2 \\ -x + 5y + 11z &= 3. \end{aligned}$$

Now let us eliminate the appearance of x from the third equation. To do this add the first equation to the third equation.

$$\begin{aligned} x - 2y - z &= 4 \\ y + 3z &= 2 \\ 3y + 10z &= 7. \end{aligned}$$

The next step is to eliminate y from the third equation by using the second equation. To do this, take the second equation, multiply it by -3 and add it to the third equation,

$$\begin{aligned} x - 2y - z &= 4 \\ y + 3z &= 2 \\ z &= 1. \end{aligned}$$

This completes the elimination step. Notice the characteristic upside down staircase shape of the equations. Now we do backwards substitution. Since the last equation does not involve either x or y, we read off the value for z from the last equation, to get $z = 1$. Now take this value for z and use the second equation, which does not involve x to solve for y,

$$y + 3 = 2.$$

This gives $y = -1$. Finally substitute the values for x and y to determine x,

$$x + 2 - 1 = 4,$$

to get $x = 3$. The solution is $(x, y, z) = (3, -1, 1) \in \mathbb{R}^3$. It is easy to check that this is indeed a solution. Strictly speaking we should also check that there are no other solutions. We will come to this point later.

Let us introduce some notation, which for the time being we will think of as just being a convenience. Instead of carrying around the variables x, y, z etc, let us just put the information into a table. We represent the system

$$\begin{aligned} x - 2y - z &= 4 \\ 2x - 3y + z &= 10 \\ -x + 5y + 11z &= 3. \end{aligned}$$

by the **augmented matrix**

$$B = \begin{pmatrix} 1 & -2 & -1 & | & 4 \\ 2 & -3 & 1 & | & 10 \\ -1 & 5 & 11 & | & 3 \end{pmatrix}.$$

The **coefficient matrix** is

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ -1 & 5 & 11 \end{pmatrix}.$$

A is a matrix with three rows and three columns, denoted 3×3. Note that the rows represent equations and the columns variables. Let

$$b = \begin{pmatrix} 4 \\ 10 \\ 3 \end{pmatrix}.$$

b is also known as a column vector. Clearly b, as a matrix, is 3×1. Then the augmented matrix is formed from the two matrices A and b

$$\begin{pmatrix} A & | & b \end{pmatrix}$$

If we let

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then we may also write down the system of equations very compactly as

$$Av = b.$$

It is possible to understand the previous method of solving linear equations in terms of the augmented matrix only. We start with,

$$\begin{pmatrix} 1 & -2 & -1 & | & 4 \\ 2 & -3 & 1 & | & 10 \\ -1 & 5 & 11 & | & 3 \end{pmatrix}.$$

The first step is to take the first row, multiply it by -2 and add it to the second row,

$$\begin{pmatrix} 1 & -2 & -1 & | & 4 \\ 0 & 1 & 3 & | & 2 \\ -1 & 5 & 11 & | & 3 \end{pmatrix}.$$

This puts a zero in the second row, first column. The next step is to take the first row, multiply it by 1 and add it to the third row,

$$\begin{pmatrix} 1 & -2 & -1 & | & 4 \\ 0 & 1 & 3 & | & 2 \\ 0 & 3 & 10 & | & 7 \end{pmatrix}.$$

This puts a zero in the third row, first column. The next step is to take the second row, multiply it by -3 and add it to the third row,

$$\begin{pmatrix} 1 & -2 & -1 & | & 4 \\ 0 & 1 & 3 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}.$$

This puts a zero in the third row, second column. At this stage, we think of this augmented matrix as representing a system of equations and use the old method of back substitution to solve this system.

Let us look at a slightly more complicated example. Suppose that we start with a system of four equations in four unknowns,

$$\begin{aligned} x + 3y - 2z - w &= -1 \\ -6x - 15y + 9z + 9w &= 9 \\ -x \qquad - z + 4w &= 5 \\ 4x + 10y - 5z - 2w &= -3. \end{aligned}$$

We first replace this by the augmented matrix,

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & -1 \\ -6 & -15 & 9 & 9 & | & 9 \\ -1 & 0 & -1 & 4 & | & 5 \\ 4 & 10 & -5 & -2 & | & -3 \end{pmatrix}.$$

The first step is to use the one in the first row, first column to eliminate the -6, -1 and 4 from the first column. To do this we add 6, 1, -4 times the first row to the second, third and fourth row, to get

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & -1 \\ 0 & 3 & -3 & 3 & | & 3 \\ 0 & 3 & -3 & 3 & | & 4 \\ 0 & -2 & 3 & 2 & | & 1 \end{pmatrix}.$$

The next step is to multiply the second row by $1/3$ to get a one in the second row, second column,

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & -1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 3 & -3 & 3 & | & 4 \\ 0 & -2 & 3 & 2 & | & 1 \end{pmatrix}.$$

Now we eliminate the 3 and the -2 in the second column. To do this we add -3 and 2 times the second row to the third and fourth row,

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & -1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 4 & | & 1 \end{pmatrix}.$$

The next step is to get a one in the third row, third column. We cannot do this by rescaling the third row. We can do it by swapping the third and fourth rows,

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & -1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 0 & 1 & 4 & | & 1 \\ 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}.$$

Consider what happens when we try to solve the resulting linear equations by back substitution. The last equation reads

$$0x + 0y + 0z + 0w = 2.$$

There are no values for x, y, z and w which work. The original system has no solutions. Now suppose we start with the system

$$\begin{aligned} x + 3y - 2z - w &= -1 \\ -6x - 15y + 9z + 9w &= 9 \\ -x \qquad - z + 4w &= 4 \\ 4x + 10y - 5z - 2w &= -5. \end{aligned}$$

The only thing we have changed is 5 to $4 = 5 - 1$. If we follow the same steps as before, we get down to the same matrix, except that the last entry is $0 = 1 - 1$ (remember at some point we swapped two rows)

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & -1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 0 & 1 & 4 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

The last equation

$$0x + 0y + 0z + 0w = 0,$$

places no restriction on x, y, z and w. The previous equation reads,

$$z + 4w = 1.$$

Using this equation, we can solve for z in terms of w, to get $z = 1 - 4w$. We can use this value for z in the second equation to determine y in terms of w,

$$y - (1 - 4w) + w = 1,$$

to get $y = 2 - 5w$. Finally, we can use this value for y and z in the first equation to solve for x,

$$x + 3(2 - 5w) - 2(1 - 4w) - w = -1,$$

to get

$$x = -5 - 8w.$$

We get the family of solutions,

$$(x, y, z, w) = (-5 + 8w, 2 - 5w, 1 - 4w, w).$$

Note that no matter the value of w, we get a solution for the original system of linear equations. For example if we pick $w = 0$, we get the solution

$$(x, y, z, w) = (-5, 2, 1, 0),$$

but if we choose the value $w = 1$, we get the solution

$$(x, y, z, w) = (3, -3, -3, 1).$$

In particular this system has infinitely many solutions. As before the reason for this is because the original system of equations was not independent. In fact even the first three equations are not independent. We can think of this as being the same thing as the first three rows are not independent. The third row minus four times the first row is the same as the second row plus the first row (we can see this by following the steps of the Gaussian elimination). This is the same as saying the third row is equal to the second row plus three times the first row. It is then automatic that any solution to the first equation and the second equations is a solution to the third equation. Note also that we can represent the solutions in a slightly different way,

$$(x, y, z, w) = (-5, 2, 1, 0) + w(8, -5, -4, 1).$$