NINTH HWK, DUE THURSDAY DECEMBER 4TH

Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

1.(10pts) Find the Jordan canonical form of the following complex matrix, where $\lambda \in \mathbb{C}$,

$$A = \begin{pmatrix} \lambda & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(*Hint: You don't need to find P.*)

Now do the case when A is a $k \times k$ matrix with λ along the main diagonal, a 1 in every entry above the main diagonal and a zero everywhere else, so that if $A = (a_{ij})$ then

$$a_{ij} = \begin{cases} \lambda & \text{if } i = j \\ 1 & \text{if } i < j \\ 0 & \text{if } i > j. \end{cases}$$

2.(10pts) If $A \in M_{n,n}(\mathbb{C})$ is in Jordan canonical form then what is the Jordan form of A^t? Why? (*Hint: First do the case when $A = B_k(\lambda)$*).

3.(5pts) If B is a nilpotent matrix, what is the connection between the number of Jordan blocks of size $k \times k$ in the Jordan canonical form of B and $\nu(B^i)$, the nullity of B^i?

4.(20pts) Let $A \in M_{n,n}(\mathbb{C})$. True or false? Give appropriate reasons.

(i) Knowing the minimal polynomial of A determines the Jordan canonical form of A.

(ii) Knowing the minimal and characteristic polynomial of A determines the Jordan canonical form of A.

(iii) If A is diagonalisable then the characteristic polynomial determines the Jordan canonical form of A.

(iv) Let V be a vector space and let P be a subset which contains zero and is closed under addition. Then P is a subspace of V.

5.(10pts) Calculate the determinant of the following real matrix,

$$\begin{vmatrix} 2 & -2 & 1 & -1 \\ 3 & -2 & 2 & 1 \\ 1 & -1 & 1 & 4 \\ 0 & 0 & 2 & -1 \end{vmatrix}.$$

6.(15pts) A sequence of numbers

$$s_0 = 0, 0, 1, 2, 5, 10, 21, 42, 85, 170, \ldots, s_n, \ldots$$

is defined recursively by the rule, $s_n = 2s_{n-1} + s_{n-2} - 2s_{n-3}$. Find a closed form expression for s_n.

Bonus Challenge Problems:

7.(10pts) What are the possible Jordan forms for a matrix $A \in M_{4,4}(\mathbb{F}_2)$? What are the corresponding minimal and characteristic polynomials?

8.(5pts) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a sequence of n scalars. Let A be the matrix whose ith column vector has entries the powers of λ_i, from 0 to $n-1$. Therefore

$$A = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix},$$

when $n = 2$ and

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix},$$

when $n = 3$.

What is $\det(A)$ and why?