EIGHTH HWK, DUE THURSDAY NOVEMBER 20TH

Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

1. (10pts) Let $\phi\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the function $\phi(r_1, r_2, r_3, r_4) = (r_1 + 2r_2 - r_3 + 3r_4, 2r_1 + 4r_2 - r_3 + 2r_4, 3r_1 + 6r_2 - 2r_3 + 5r_4)$.
(i) Show that ϕ is linear. What is the matrix corresponding to ϕ?
(ii) Find linear isomorphisms $f\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ and $g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\psi = g \circ \phi \circ f^{-1}\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ has the form $\psi(r_1, r_2, r_3, r_4) = (r_1, r_2, \dots, r_k, 0, \dots, 0)$ given by Theorem 13.1.

2. (10 pts) (i) Let $f(x) \in P(F)$. Let $\alpha \in F$. Show that the polynomial $x - \alpha$ is a factor of $f(x)$ (that if $f(x) = (x - \alpha)g(x)$ for some $g(x) \in P(F)$) if and only if α is a root of $f(x)$ (that is a solution to the equation $f(x) = 0$).
(ii) Suppose that $f(x) = g(x)h(x)$ factors, where $f(x)$, $g(x)$ and $h(x) \in P(F)$. Show that the degree of $f(x)$ is the sum of the degrees of $h(x)$ and $g(x)$.

3. (15 pts) Say that a non-zero polynomial $f(x) \in P(F)$ is **reducible** if it factors as $f(x) = g(x)h(x)$, where $g(x)$ and $h(x) \in P(F)$ have degree greater than zero. If $f(x)$ is not reducible then we say that $f(x)$ is **irreducible**.
(i) Suppose that $f(x)$ has degree two or three. Show that $f(x)$ is reducible if and only if the equation $f(x) = 0$ has a root.
(ii) Find all irreducible monic polynomials of degree two in $P(\mathbb{F}_3)$.
(iii) Find all irreducible polynomials of degree three in $P(\mathbb{F}_2)$.

4. (20pts) Are the following statements true or false? Give appropriate reasons.
(i) Every irreducible real polynomial has degree one.
(ii) Every diagonalisable matrix is invertible.
(iii) Suppose that A and $B \in M_{n,n}(F)$ are two similar matrices. If A is diagonalisable then so is B.
(iv) Let $A \in M_{n,n}(\mathbb{R})$. If A is diagonalisable as a complex matrix then A is diagonalisable as a real matrix.

Bonus Challenge Problems:

5. (10 pts) Find all irreducible polynomials of degree four with coefficients in $\mathbb{F}_2$.