

6. FIBRE PRODUCTS

We start with some basic properties of schemes.

Definition 6.1. We say that a scheme is **connected** (respectively **irreducible**) if its topological space is connected (respectively irreducible).

Definition 6.2. We say that a scheme is **reduced** if $\mathcal{O}_X(U)$ contains no nilpotent elements, for every open set U .

Remark 6.3. It is straightforward to prove that a scheme is reduced iff the stalk of the structure sheaf at every point contains no nilpotent elements.

Definition 6.4. We say that a scheme X is **integral** if for every open set $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.

Proposition 6.5. A scheme X is integral iff it is irreducible and reduced.

Proof. Suppose that X is integral. Then X is surely reduced. Suppose that X is reducible. Then we can find two non-empty disjoint open sets U and V . But then

$$\mathcal{O}_X(U \cup V) \simeq \mathcal{O}_X(U) \times \mathcal{O}_X(V),$$

which is surely not an integral domain.

Now suppose that X is reduced and irreducible. Let $U \subset X$ be an open set and suppose that we have f and $g \in \mathcal{O}_X(U)$ such that $fg = 0$. Set

$$Y = \{x \in U \mid f_x \in m_x\} \quad \text{and} \quad Z = \{x \in U \mid g_x \in m_x\}.$$

Then Y and Z are both closed and by assumption $Y \cup Z = U$. As X is irreducible, one of Y and Z is the whole of U , say Y . We may assume that $U = \text{Spec } A$ is affine. But then $f \in A$ belongs to the intersection of all the prime ideals of A , which is the zero ideal, as A contains no nilpotent elements. \square

Definition 6.6. We say that a scheme X is **locally Noetherian**, if there is an open affine cover, such that the corresponding rings are Noetherian. If in addition the topological space is compact, then we say that X is **Noetherian**.

Remark 6.7. There are examples of schemes whose topological space is Noetherian which are not Noetherian schemes.

A key issue in this definition is whether or not we can replace an open cover, by every affine cover.

Proposition 6.8. *A scheme X is locally Noetherian iff for every open affine $U = \text{Spec } A$, A is a Noetherian ring.*

Proof. It suffices to prove that if X is locally Noetherian, and $U = \text{Spec } A$ is an open affine subset then A is a Noetherian ring. Now if B is a Noetherian ring, then so is any localisation B_f . But the open sets $U_f = \text{Spec } B_f$ form a base for the topology on $\text{Spec } B$, and so it follows that on a locally Noetherian scheme, there is a base of open affine sets, which are the spectra of Noetherian rings.

Thus we have reduced to proving that if $X = \text{Spec } A$ is an affine scheme which can be covered by open affine schemes, which are the spectra of Noetherian rings, then A is Noetherian. Let $U = \text{Spec } B$, with B a Noetherian ring. Then there is an element $f \in A$ such that $U_f \subset U$. Let g be the image of f in B . Then $A_f \simeq B_g$, whence A_f is Noetherian. So we can cover X by open subsets $U_f = \text{Spec } A_f$, with A_f Noetherian. As X is compact, we may assume that we have a finite cover. Now apply (6.9). \square

Lemma 6.9. *Let A be a ring, and let f_1, f_2, \dots, f_r be elements of A which generate the unit ideal.*

If A_{f_i} is Noetherian, for $1 \leq i \leq r$ then so is A .

Proof. Suppose that we have an ascending chain of ideals,

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \dots,$$

of A . Then for each i ,

$$\phi_i(\mathfrak{a}_1) \cdot A_{f_i} \subset \phi_i(\mathfrak{a}_2) \cdot A_{f_i} \dots,$$

is an ascending chain of ideals inside A_{f_i} , where $\phi_i: A \rightarrow A_{f_i}$ is then natural map. As each A_{f_i} is Noetherian, all of these chains stabilise. But then the first chain stabilises, by (6.10). \square

Lemma 6.10. *Let A be a ring, and let f_1, f_2, \dots, f_r be elements of A which generate the unit ideal. Suppose that \mathfrak{a} is an ideal and let $\phi_i: A \rightarrow A_{f_i}$ be the natural maps. Then*

$$\mathfrak{a} = \bigcap_{i=1}^r \phi_i^{-1}(\phi_i(\mathfrak{a}) \cdot A_{f_i}).$$

Proof. The fact that the LHS is included in the RHS is clear. Conversely suppose that b is an element of the RHS. In this case

$$\phi_i(b) = \frac{a_i}{f_i^{n_i}},$$

for some $a_i \in \mathfrak{a}$ and some positive integer n_i . As there are only finitely many indices, we may assume that $n = n_i$ is fixed. But then

$$f^{m_i}(f^n b - a) = 0,$$

for $1 \leq i \leq r$. Once again, we may assume that $m = m_i$ is fixed. It follows that $f_i^N b \in \mathfrak{a}$, for $1 \leq i \leq r$, where $N = n + m$. But since f_1, f_2, \dots, f_r generate the unit ideal, then so does their N th power. Hence we may write

$$1 = \sum_i c_i f_i^N.$$

But then

$$b = \sum_i c_i f_i^N b \in \mathfrak{a}.$$

□

Definition 6.11. A morphism $f: X \rightarrow Y$ is **locally of finite type** if there is an open affine cover $V_i = \text{Spec } B_i$ of Y , such that $f^{-1}(V_i)$ is a union of affine sets $U_{ij} = \text{Spec } A_{ij}$, where each A_{ij} is a finitely generated B_i -algebra. If in addition, we can take U_{ij} to be a finite cover of $f^{-1}(V_i)$, then we say that f is of **finite type**.

Definition 6.12. We say that a morphism $f: X \rightarrow Y$ is **finite** if we may cover Y by open affines $V_i = \text{Spec } B_i$, such that $f^{-1}(V_i) = \text{Spec } A_i$ is an affine set, where A_i is a finitely generated B_i -module.

In both cases, it is easy to prove that we can take V_i to be any affine subset of Y .

Example 6.13. Let

$$f: \mathbb{A}_k^1 - \{0\} \rightarrow \mathbb{A}_k^1$$

by the natural map given by the natural localisation map

$$k[x] \rightarrow k[x]_x$$

As an algebra over $k[x]$, the ring $k[x]_x \simeq k[x, x^{-1}]$ is generated by x^{-1} , so that f is of finite type. However the $k[x]$ -module $k[x, x^{-1}]$ is not finitely generated (there is no way to generate all the negative powers of x), so that f is not finite.

Example 6.14. If V is an irreducible variety over an algebraically closed field, then $t(V)$ is an integral variety of finite type over $\text{Spec } k$.

Definition 6.15. A **variety** is any reduced scheme of finite type over an algebraically closed field.

There are varieties in the sense of (6.15) which do not correspond to quasi-projective varieties.

Definition 6.16. Let X be a scheme and U an open subset of X . Then the pair $(U, \mathcal{O}_X|_U)$ is a scheme, which is called an **open subscheme** of X . An **open immersion** is a morphism $f: X \rightarrow Y$ which induces an isomorphism of X with an open subset of Y .

Definition 6.17. A **closed immersion** is a morphism of schemes $\phi = (f, f^\#): Y \rightarrow X$ such that f induces a homeomorphism of Y with a closed subset of X and furthermore the map $f^\#: \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is surjective. A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say that two closed immersions $f: Y \rightarrow X$ and $f': Y' \rightarrow X$ are equivalent if there is an isomorphism $i: Y' \rightarrow Y$ such that $f' = f \circ i$.

Despite the seemingly tricky nature of the definition of a closed immersion, in fact it is easy to give examples of closed subschemes of an affine variety.

Lemma 6.18. Let A be a ring and let \mathfrak{a} be an ideal of A . Let $X = \text{Spec } A$ and $Y = \text{Spec } A/\mathfrak{a}$.

Then Y is a closed subscheme of X .

Proof. The quotient map $\text{map } A \rightarrow A/\mathfrak{a}$ certainly induces a morphism of schemes $\phi: Y \rightarrow X$. Indeed f is certainly a homeomorphism of Y with $V(\mathfrak{a})$ and $f^\#: \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is surjective as the map on stalks is induced by the quotient map, which is surjective. \square

In fact, it turns out that every closed subscheme of an affine scheme is of this form. It is interesting to look at some examples.

Example 6.19. Let $X = \mathbb{A}_k^2$. First consider $\mathfrak{a} = \langle y^2 \rangle$. The support of Y is the x -axis. However the scheme Y is not reduced, even though it is irreducible. It is clear from this example that in general there are many closed subschemes with the same support (equivalently there are many ideals with the same radical). Now consider the ideal $\langle x^2, xy, y^2 \rangle$, the double of the maximal ideal of a point. Similarly consider $\langle x, y^2 \rangle$. Finally consider $\langle x^2, xy \rangle$. The support of this ideal is the y -axis. But this time the only local ring which has nilpotents is the local ring of the origin. We call the origin an **embedded point**.

Definition 6.20. Let V be an irreducible affine variety and let W be a closed irreducible subvariety, defined by the prime ideal \mathfrak{p} . Let $X = t(V)$ and $Y = t(W)$. Then $X = \text{Spec } A$ and Y is defined by \mathfrak{p} . The **n th infinitesimal neighbourhood of Y in X** , denoted Y_n , is the closed subscheme of X corresponding to \mathfrak{p}^n .

Note that the n th infinitesimal neighbourhood of Y in X is a closed subscheme whose support coincides with Y , but whose structure sheaf

contains lots of nilpotent elements. As the name might suggest, Y_n carries more information about how Y sits inside X , than does Y itself.

Definition-Lemma 6.21. *Let X be scheme and let Y be a closed subset. Then Y has a unique reduced subscheme structure, called the **reduced induced subscheme structure**.*

Proof. We first assume that $X = \text{Spec } A$ is affine. Let \mathfrak{a} be the ideal obtained by intersecting all the prime ideals in Y . Then \mathfrak{a} is the largest ideal for which $V(\mathfrak{a}) = Y$.

Now suppose that X is an arbitrary scheme. For each open affine subset $U_i \subset X$, let $Y_i \subset U_i$ be the reduced induced subscheme structure on $Y \cap U_i$. I claim that the restriction to $Y_i \cap Y_j$ of the two scheme structures on Y_i and Y_j are isomorphic, by an isomorphism which agrees on triple intersections. Using this, it is a straightforward result in sheaf theory to obtain a sheaf \mathcal{O}_Y on Y .

It is not hard to reduce to the case where $U = \text{Spec } A$, $V = \text{Spec } A_f$ and the restriction of the reduced induced subscheme structure to V is the same as the reduced induced subscheme structure on V . But this is the same as to say that if \mathfrak{a} is the intersection of those prime ideals of A which are contained in Y , then $\mathfrak{a}A_f$ is the intersection of those prime ideals of A_f which are contained in Y , which is clear. \square

Note that if a scheme X has a topological space with one point, then X must be affine, and the stalk of the structure sheaf at the unique point completely determines X , and this ring have exactly one prime ideal. Moreover a morphism of X into another scheme Y , is equivalent to picking a point y of Y and a morphism of local rings

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

But to give a morphism of local rings is the same as to give an inclusion of the quotients of the maximal ideals. Thus to give a morphism of $X = \{x\}$ into Y , sending x to y , we need to specify an inclusion of the residue field of x into the residue field of y .

The main result of this section is

Theorem 6.22. *The category of schemes admits fibre products.*

A key part of the proof is to pass from the local case (in which case all three schemes are affine) to the global case. To do this, we need to be able to construct morphisms, by constructing them locally. We will need:

Theorem 6.23. *Let $f_i: U_i \longrightarrow Y$ be a collection of morphisms of schemes, with a varying domain, but a fixed target.*

Suppose that for each pair of indices i and j , we are given open subsets $U_{ij} \subset U_i$, and isomorphisms $\phi_{ij}: U_i \rightarrow U_j$, such that $f_i|_{U_{ij}} = f_j \circ \phi_{ij}: U_{ij} \rightarrow Y$, where the morphisms ϕ_{ij} satisfy two patching conditions:

- (1) ϕ_{ii} is the identity,
- (2) For all i, j and k ,

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the triple intersection U_{ijk} .

(Note that in particular $\phi_{ij}^{-1} = \phi_{ji}$.)

Then there is a morphism of schemes $f: X \rightarrow Y$, an open cover of X by open sets X_i and isomorphisms $\psi_i: U_i \rightarrow X_i$, such that $f_i = f \circ \psi_i: U_i \rightarrow Y$ and $\psi_i|_{U_{ij}} = \psi_j \circ \phi_{ij}: U_{ij} \rightarrow Y$.

X is unique, up to unique isomorphism, with these properties.

We prove (6.23) in two steps (one of which can be further broken down into two substeps):

- Construct the scheme X .
- Construct the morphism f .

In fact, having constructed X , it is straightforward to construct f . Since a scheme consists of two parts, a topological space and a sheaf, we can break the first step into two smaller pieces:

- Construct the underlying topological space.
- Construct the structure sheaf.

We first show how to patch a sheaf, which is the hardest part:

Lemma 6.24. *Let X be a topological space, and let $\{X_i\}$ be an open cover of X . Suppose that we are given sheaves \mathcal{F}_i on X_i and for each i and j an isomorphism*

$$\phi_{i,j}: \mathcal{F}_i|_{X_{ij}} \rightarrow \mathcal{F}_j|_{X_{ij}}$$

We suppose further that our isomorphisms satisfy two conditions:

- (1) ϕ_{ii} is the identity,
- (2) For all i, j and k ,

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the triple intersection X_{ijk} .

Then there is a sheaf \mathcal{F} on X , together with isomorphisms, $\psi_i: \mathcal{F}|_{U_i} \rightarrow \mathcal{F}_i$, which satisfy $\psi_j = \phi_{ij} \circ \psi_i$. Further \mathcal{F} is unique up to unique isomorphism, with these properties.

Proof. We just show how to define \mathcal{F} and leave the rest to the interested reader. Let $U \subset X$ be any open set, and let $U_i = U \cap X_i$.

$$\mathcal{F}(U) = \left\{ (s_i) \in \bigoplus_i \mathcal{F}(U_i) \mid \phi_{ij}(s_i|_{U_{ij}}) = s_j|_{U_{ji}} \right\}. \quad \square$$

The next step is to bump this up to schemes:

Lemma 6.25. *Suppose that we are given schemes U_i , and subschemes $U_{ij} \subset U_i$, together with isomorphisms,*

$$\phi_{ij}: U_{ij} \longrightarrow U_{ji},$$

which satisfy:

- (1) ϕ_{ii} is the identity,
- (2) For all i, j and k ,

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the triple intersection U_{ijk} .

Then there is a scheme X , together with open sets X_i and isomorphisms $\psi_i: U_i \longrightarrow X_i$ which satisfy $\psi_i|_{U_{ij}} = \phi_{ij} \circ \psi_i: U_{ij} \longrightarrow X$.

Proof. We first construct the topological space X . Let

$$X = \coprod_i U_i / \sim \quad \text{where} \quad x_i \in U_{ij} \sim x_j \in U_{ji} \text{ iff } \phi_{ij}(x_i) = x_j.$$

Here \sim denotes the equivalence relation generated by the rule on the RHS, and X is just the quotient topological space (which always exists). Note that $X_i = U_i / \sim \subset X$ is an open subset of X and there are homeomorphisms $\phi_i: U_i \longrightarrow X_i$. Now construct a sheaf \mathcal{O}_X on X , using (6.24). This gives us a locally ringed space (X, \mathcal{O}_X) and the remaining properties can be easily checked. \square

Proof of (6.23). Let X be the scheme constructed in (6.25). But to give a morphism $f: X \longrightarrow Y$ is the same as to give morphisms $f_i: X_i \longrightarrow Y$, compatible on overlaps. \square

There are a couple of interesting examples of the construction of schemes. The first is to take U_{ij} empty (so that there are no patching conditions at all). The resulting scheme is called the disjoint union and is denoted

$$\coprod_i X_i.$$

Another more interesting one proceeds as follows. Take two copies U_1 and U_2 of the affine line. Let $U_{12} = U_{21}$ be the complement of the origin, and let ϕ_{12} be the identity. Then X is obtained by identifying every point, except the origin. Note that this is like the classical construction

of a topological space, which is locally a manifold, but which is not Hausdorff. Of course no scheme is ever Hausdorff (apart from the most trivial examples) and it turns out that there is an appropriate condition to express the Hausdorff condition for schemes, which we shall see later.

Finally we turn to the problem of glueing morphisms. Suppose that we are given two schemes X and Y . Then to give a morphism $f: X \rightarrow Y$ is the same as to give a collection of morphisms $f_i: U_i \rightarrow Y$, such that f_i and f_j restrict to the same morphism on U_{ij} .

Proof of (6.22). Let X and Y be two schemes over S . We want to construct the fibre product.

First suppose that $X = \text{Spec } A$, $Y = \text{Spec } B$ and $S = \text{Spec } R$. Then there are ring homomorphisms $R \rightarrow A$ and $R \rightarrow B$ and so A and B are R -algebras. As $C = A \otimes_R B$ is the pushout in the category of rings, it follows that $Z = \text{Spec } C$ is the fibre product in the category of affine schemes.

Note that if $X \times_S Y$ is the fibre product of X and Y over S and U is an open subset of X , then $p_1^{-1}(U) \times_S Y$ is a fibre product for U and Y over S . Indeed, suppose that Z maps to U and Y . Then Z maps to X and Y , whence to the fibre product. By definition of the fibre product, Z lies over U under projection down to X . But then Z lands in $p_1^{-1}(U) \times_S Y$.

Note that if U_i is an open cover of X and the fibre product $U_i \times_S Y$ exists, then since both the restriction of $U_i \times_S Y$ and $U_j \times_S Y$ to $p_1^{-1}U_{ij}$ are fibre products, there are natural isomorphisms on overlaps, satisfying conditions (1) and (2) above. It follows that we may patch these schemes together to obtain a scheme which we denote $X \times_S Y$.

The universal property follows, since we can patch morphisms.

Since an arbitrary scheme X can be covered by open affines, it follows that we can form the fibre product $X \times_S Y$ whenever Y and S are affine. Similarly, switching X and Y , it follows that we can form the fibre product, whenever S is affine.

Now take an affine cover S_i of S . Let X_i and Y_i be the inverse image of S_i (meaning take the open subscheme on the open set $p_j^{-1}(S_i)$). Then the fibre product $X_i \times_S Y_i$ exists. But in fact this is also a fibre product for $X_i \times_S Y$, since anything lying over X_i automatically lies over Y_i . Since X_i forms an open cover of X we are done. \square

It turns out that the fibre product is extremely useful.

Definition 6.26. Let $f: X \rightarrow S$ be a morphism of schemes, and let $s \in S$ be a point of S . The **fibre over s** is the fibre product over the morphism f and the inclusion of s in S , where the point s is given a scheme structure by taking the residue field $\kappa(s)$.

It is interesting to see what happens in some specific examples. First consider a family of conics in the plane,

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle ty - x^2 \rangle}.$$

The inclusion

$$k[t] \rightarrow \frac{k[x, y, t]}{\langle ty - x^2 \rangle},$$

realises X as a family over the affine line over k ,

$$f: X \rightarrow \mathbb{A}_k^1.$$

Pick a point $p \in \mathbb{A}^1$. If the point is maximal, this is the same as picking a scalar, and of course the residue field is nothing more than k . If we pick a non-zero scalar a , then we just get the conic defined by $ay - x^2$ in $k[x, y]$ (since tensoring by k won't change anything),

$$X_p = \operatorname{Spec} \frac{k[x, y]}{\langle ay - x^2 \rangle}.$$

But now suppose that $a = 0$. In this case the above reduces to

$$X_0 = \operatorname{Spec} \frac{k[x, y]}{\langle x^2 \rangle},$$

a double line. It is also interesting to consider the fibre over the generic point ξ , corresponding to the maximal ideal $\langle 0 \rangle$. In this case the residue field is $k(t)$, and the **generic fibre** is

$$X_\xi = \operatorname{Spec} \frac{k(t)[x, y]}{\langle ty - x^2 \rangle},$$

which is the conic $V(ty - x^2) \subset \mathbb{A}_{k(t)}^2$ over the field $k(t)$.

Similarly, if we pick the family

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle xy - t \rangle}.$$

then, for $a \neq 0$, the fibre is a smooth conic, but for $t = 0$ the fibre is a pair of lines.

Once again, the point is that there are some more exotic examples, which can be treated in the same fashion. Consider for example $\operatorname{Spec} \mathbb{Z}[x]$. Once again this is a scheme over $\operatorname{Spec} \mathbb{Z}$, and once again it is interesting to compute the fibres. Suppose first that we take the

generic point. Then this has residue field \mathbb{Q} . If we tensor $\mathbb{Z}[x]$ by \mathbb{Q} , then we get $\mathbb{Q}[x]$. If we take spec of this, we get the affine line over \mathbb{Q} . Now suppose that we take a maximal ideal $\langle p \rangle$. In this case the residue field is \mathbb{F}_p the finite field with p elements. Tensoring by this field we get $\mathbb{F}_p[x]$ and taking spec we get the affine line over the finite field with p elements.

It is also possible to figure out all the prime ideals in $\mathbb{Z}[x]$. They are

- (1) $\langle 0 \rangle$
- (2) $\langle p \rangle$, p a prime number.
- (3) $\langle f(x) \rangle$, $f(x)$ irreducible over \mathbb{Q} , with content one,
- (4) maximal ideals of the form $\langle p, f(x) \rangle$, where $f(x)$ is a monic polynomial whose reduction modulo p is irreducible.

Note that the zero ideal is the generic point, and the closure of the ideal $\langle p \rangle$ is the fibre over the same ideal downstairs. The closure of an ideal of type (3) is perhaps the most interesting. It will consist of all maximal points $\langle p, g \rangle$, where g is a factor of f inside $\overline{\mathbb{F}_p}$.

It is now possible to consider closed subschemes of $\mathbb{A}_{\mathbb{Z}}^1$. For example consider

$$X = \text{Spec} \frac{\mathbb{Z}[x]}{\langle 3x - 16 \rangle}.$$

Fibre by fibre, we get a collection of subschemes of $\mathbb{A}_{\mathbb{F}_p}^1$. If we reduce modulo 5, that is tensor by \mathbb{F}_5 , then we get

$$X = \text{Spec} \frac{\mathbb{F}_5[x]}{\langle 3x - 1 \rangle},$$

a single point. However something strange happens over the prime 3, since we get an equation which cannot be satisfied. If we think of this as the graph of the rational map $16/3$, then we have a pole at 3, which cannot be removed. Of course over 2, this rational function is zero.

Now suppose that we consider $x^2 - 3$. Then we get a conic. In fact, this is the same as considering

$$\frac{\mathbb{Z}[x]}{\langle x^2 - 3 \rangle} = \mathbb{Z}[\sqrt{3}].$$

So the seemingly strange picture we had before becomes a little more clear. Now suppose that we consider a plane conic in $\mathbb{A}_{\mathbb{Z}}^2$,

$$X = \text{Spec} \frac{\mathbb{Z}[x, y]}{\langle x^2 - y^2 - 5 \rangle}.$$

Over the typical prime, we get a smooth conic in the corresponding affine plane over a finite field. But now consider what happens over $\langle 2 \rangle$

and $\langle 5 \rangle$. Modulo two, we have

$$x^2 - y^2 - 5 = (x + y + 1)^2,$$

and modulo 5 we have

$$x^2 - ty^2 - 5 = (x - y)(x + y).$$

Thus we get a double line over $\langle 2 \rangle$ and a pair of lines over $\langle 5 \rangle$.

Let us return to the case of $x^2 - 3$, and consider the residue fields. Recall that there are three cases.

- (1) If p divides the discriminant of K/\mathbb{Q} (which in this case is 12), that is $p = 2$ or 3 , then the ideal $\langle p \rangle$ is a square in A .

$$\langle 2 \rangle A = (\langle 1 + \sqrt{3} \rangle)^2,$$

and

$$\langle 3 \rangle A = (\langle \sqrt{3} \rangle)^2.$$

- (2) If 3 is a square modulo p , the prime $\langle p \rangle$ factors into a product of distinct primes,

$$\langle 11 \rangle A = \langle 4 + 3\sqrt{3} \rangle \langle 4 - 3\sqrt{3} \rangle,$$

or

$$\langle 13 \rangle A = \langle 4 + \sqrt{3} \rangle \langle 4 - \sqrt{3} \rangle,$$

- (3) If $p > 3$ and 3 is not a square mod p (e.g $p = 5$ and 7), the ideal $\langle p \rangle$ is prime in A .

Let us consider the stalks and residue fields in all three cases. In the first case we get

$$A/\mathfrak{p}^2,$$

and the residue field is \mathbb{F}_p . In the second case there are two points with coordinate rings \mathbb{F}_p . Finally in the third case there is a single point with coordinate ring

$$\mathbb{F}_p^2,$$

the unique finite field with p^2 elements. Note that in all three cases, the coordinate ring of the inverse image has length two over the coordinate ring of the base (in our case \mathbb{F}_p). In fact this is the general picture. Finite maps have a degree, and the length of the coordinate ring over the base is equal to this degree.

Another useful way to think of the fibre product, is as a base change. In arithmetic, one always wants to compare what happens over different fields, or even different rings.

Now consider an interesting example over non-algebraically close fields. Consider the inclusion $\mathbb{R} \longrightarrow \mathbb{C}$. This gives a morphism of schemes,

$$f: X \longrightarrow Y,$$

where X and Y are schemes with only one point, but the first has coordinate ring \mathbb{C} and the second \mathbb{R} . Now consider what happens when we make the base change f over f . Then we get a scheme

$$X \times_Y X.$$

Note that this has degree two over X . Since \mathbb{C} is algebraically closed, in fact this must consist of two points, even though f only has one point in the fibre. Algebraically,

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^2.$$

In particular, the property of being irreducible is not preserved by base change. Consider also the example of $x^2 - t$ over the affine line, with coordinate t , say over an algebraically closed field. Then the fibre over every closed point, except zero, is reducible. But the fibre over the generic point is irreducible, since $x^2 - t$ won't factor. However suppose that we make a base change of the affine line by the affine line given by

$$\mathbb{A}_k^1 \longrightarrow \mathbb{A}_k^1 \quad \text{given by} \quad t \longrightarrow t^2.$$

After base change, the new scheme is given by $x^2 - t^2$. But this factors, even over the generic point

$$x^2 - t^2 = (x - t)(x + t).$$

Definition 6.27. *Let P be a property of schemes. We say that a morphism $f: X \longrightarrow S$ is **universally** P if the fibres of f have property P , after any base change $S' \longrightarrow S$.*

In particular, we say that f is **universally irreducible**, if every fibre of f is irreducible, after any base change. It turns out the locus in the base where a morphism is universally irreducible is open, even though the same is not true for irreducible.