1. Why schemes?

Schemes were introduced by Grothendieck more than fifty years ago into the world of algebraic geometry. In much the same way as measure theory, nearly everyone in the field almost immediately adopted the new definitions. But like measure theory for someone on the outside the whole theory seems remarkably abstract and hard to absorb. For this reason it might help to quickly review the reasons why schemes were introduced in the first place. Then in the course of these lectures we will see how the theory of schemes deals with the limitations of working with varieties.

Geometrically there are three compelling reasons to work with more general objects than varieties. Firstly, it is desirable to have a definition of an algebraic variety which is independent of any embedding into projective space. Compare this with the definition of a group. Originally mathematicians thought of groups as subsets of the set of permutations of a set, which are closed under composition and inverses. It is clearly much better to have the abstract definition of a group and then consider all the possible ways of embedding the group into permutation groups. This way one can think about groups being isomorphic, without worrying about a particular embedding. Similarly one of the big conceptual advances of the twentieth century was a definition of an abstract manifold. Notice however that the case of varieties is much harder, since the Zariski topology is so weak.

Secondly, it is easy to give examples of varieties with an action of a finite group, such that the quotient is not a variety. On the other hand, locally the quotient is an algebraic variety, so that the quotient ought to be very close to a variety.

Thirdly in the study of families of algebraic varieties, it is clear that some fibres are not really varieties at all. As a concrete example, consider the case of conics in $\mathbb{P}^2$. If coordinates are $[X : Y : Z]$ on $\mathbb{P}^2$ then any conic in $\mathbb{P}^2$ is given by

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY = 0,$$

and so the set of all conics is very naturally represented by $\mathbb{P}^5$, with coordinates $[a : b : c : d : e : f]$. In fact, there is then a universal family

$$V \subset \mathbb{P}^2 \times \mathbb{P}^5,$$

where is the closed subset of $\mathbb{P}^2 \times \mathbb{P}^5$ given by the bihomogeneous polynomial

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY = 0.$$

But then there are some very strange conics. For example, the conic $X^2 = 0$. Simply as a set, we get the straight line $X = 0$, which is clearly not really the right way to think about the conic $X^2 = 0$. For example, consider the family of conics $X^2 - t = 0$. When $t = 0$ we definitely have a conic curve, but when the zero set is only a straight line.

Moreover there are other equally compelling reasons to enlarge the category of varieties, coming from other areas of mathematics. Suppose that we want to understand the equation

$$x^n + y^n = z^n.$$

In terms of arithmetic, we are interested in those 3-tuples (x, y, z), where x, y and $z \in \mathbb{Z}$. It is well known that determining the integral solutions is very hard, and it is natural to attack such problems by considering what happens over $\mathbb{C}$ and also what happens when we reduce modulo p, which are both considerably easier and shed light on what happens over the integers. In these terms, it seems that we have a single object X (determined by the equation) and we seek to understand X, by computing what happens when we look at the set

$$X(R) = \{\, (x, y, z) \in R^3 \mid x^n + y^n = z^n \,\},$$

where R is a commutative ring. Note also in this context, that even over a field K, it is not enough to work with zero sets over the field. For example consider the field $\mathbb{R}$. Then the family of curves

$$x^2 + y^2 = t,$$

inside $\mathbb{R}^2$, where $t \in \mathbb{R}$, is not well behaved. For $t > 0$, we get a circle, for $t = 0$ we get a single point and for $t < 0$, we get the empty set. In other words, if we have an algebraic variety, it is not enough to consider the ordinary points over $\mathbb{R}$. This becomes even clearer if we work over a finite field. It is clear that different geometric objects, which have very different dimensions, will have the same zero set.

Finally, it is often useful to attack problems in commutative algebra, by considering the corresponding affine variety. In these terms, restricting to finitedly generated algebras without nilpotents is unnecessarily restrictive.