7. Classification of Surfaces

The key to the classification of surfaces is the behaviour of the canonical divisor.

Definition 7.1. *We say that a smooth projective surface is* ***minimal*** *if K_S is nef.*

Warning: This is not the classical definition of a minimal surface.

Definition 7.2. *Let S be a smooth projective surface. We say that a curve $C \subset S$ is a* -1***-curve*** *if*

$$K_S \cdot C = C^2 = -1.$$

Theorem 7.3 (Cone Theorem)**.** *Let S be a smooth projective surface.*

Then there are countably many extremal rays $R_1, R_2, \dots$ of the closed cone of curves of S on which K_S is negative, such that

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{NE}}(S)_{K_X \geq 0} + \sum R_i.$$

Further, if $R = R_i$ is any one of these K_S-extremal rays then there is a birational morphism $\pi \colon S \longrightarrow Z$ which contracts a curve C iff C spans the ray R. There are three possibilities:

(1) *Z is a point and $S \simeq \mathbb{P}^2$.*
(2) *$\pi \colon S \longrightarrow Z$ is a $\mathbb{P}^1$-bundle over a smooth curve Z.*
(3) *$\pi \colon S \longrightarrow Z$ is a birational morphism contracting a -1-curve C, where Z is a smooth surface.*

In particular the relative Picard number of π is one, each extremal ray R_i is spanned by a rational curve and if H is any ample divisor, there are only finitely many extremal rays R_i such that $(K_X + H) \cdot R < 0$.

Remark 7.4. *The last two statements are sometimes informally stated as saying that the closed cone of curves is locally rational polyhedral on the K_S-negative side of the cone.*

Theorem 7.5 (Castelnuovo)**.** *Let S be a smooth projective surface and let $C \subset S$ be a curve.*

Then C is a -1-curve iff there is birational morphism $\pi \colon S \longrightarrow T$, which blows up a smooth point $p \in T$, with exceptional divisor C.

Theorem 7.6 (Abundance)**.** *Let S be a smooth projective surface.*

Then K_S is nef iff K_S is semiample.

Theorem 7.7 (Kodaira-Enriques)**.** *Let T be a smooth projective surface, with invariants $\kappa = \kappa(T)$, $p_g = p_q(T)$ and $q = q(T)$. Then T is birational to a surface S which falls into the following list:*

$\kappa = -\infty$**:**

Ruled surface $S \simeq \mathbb{P}^1 \times B$ *where B is a smooth curve of genus.* $q(S) = g(B)$.

$\kappa = 0$**:**

Abelian surface $p_g = 1$, $q = 2$. $S \simeq \mathbb{C}^2/\Lambda$.

Bielliptic $p_g = 0$, $q = 1$. *There is a Galois cover of $\tilde{S} \longrightarrow S$ of order at most 12 such that $\tilde{S} \simeq E \times F$, where E and F are elliptic curves.*

K3 surface $p_g = 1$, $q = 0$.

Enriques surface $p_g = 0$, $q = 0$. *There is an étale cover $\tilde{S} \longrightarrow S$ of order two, such that $\tilde{S}$ is a K3-surface.*

$\kappa = 1$**:**

Elliptic fibration *there is a contraction morphism $\pi \colon S \longrightarrow B$ with general fibre a smooth curve of genus one. $P_m(S) > 0$ for all m divisible by 12.*

$\kappa = 2$**:**

General type *ϕ_m is birational for all $m \geq 5$. In particular $P_m(X) > 0$ for all $m \geq 5$.*

In particular $\kappa \geq 0$ iff $P_{12} \geq 0$.

Definition 7.8. *Let X be a normal projective variety, let D be a nef divisor and let E be any divisor. The **nef threshold** is the largest multiple of E we can add to D, whilst preserving the nef condition:*

$$\lambda = \sup\{\, t \in \mathbb{R} \,|\, D + tE \text{ is nef}\}.$$

Definition 7.9. *Let X be a projective scheme and let D be a nef divisor. The **numerical dimension** $\nu(X, D)$ of D is the largest positive integer such that $D^k \cdot H^{n-k} > 0$, where H is an ample divisor.*

Note that if D is semiample then $\kappa(D) = \nu(D)$. We will need the following easy:

Lemma 7.10. *Let X be a normal projective variety and let D be a nef $\mathbb{Q}$-Cartier divisor.*

(1) *If $\nu(D) = 0$ then D is semiample iff $\kappa(D) = 0$.*
(2) *If $\nu(D) = 1$ then D is semiample iff $h^0(X, \mathcal{O}_X(mD)) \geq 2$, for some $m > 0$.*

In particular if $\nu(D) \leq 1$ then D is semiample iff $\nu(D) = \kappa(D)$.

Proof. Suppose that $\nu(D) = 0$. Then D is numerically trivial and it is semiample iff it is torsion. As $\kappa(D) = 0$, $D \sim_{\mathbb{Q}} B \geq 0$ and since B is numerically trivial, in fact $B = 0$.

Suppose that $\nu(D) = 1$. Pick m so that $|mD|$ contains a pencil. Let $B_i \in |mD|$, $B_1 \neq B_2$. Then $B_1 \cap B_2 = \varnothing$, since otherwise $D^2 \cdot H^{n-2} = B_1 \cdot B_2 \cdot H^{n-2} = 0$. But then $|mD|$ is base point free. ☐

Definition 7.11. *Let $\pi \colon X \longrightarrow U$ be a projective morphism.*

*The **relative cone of curves** is the cone generated by the classes of all curves contracted by π,*

$$\overline{\mathrm{NE}}(X/U) = \{\, \alpha \in \mathrm{NE}(X) \,|\, \pi_* \alpha = 0 \,\}.$$

*We say that a $\mathbb{Q}$-Cartier divisor H is π**-ample** (aka **relatively ample**, aka **ample over** U) if mH is relatively very ample (that is there is an embedding i of X into $\mathbb{P}^n_U = \mathbb{P}^n \times U$ over U such that $\mathcal{O}_X(mH) = i^*\mathcal{O}(1)$).*

We say that an $\mathbb{R}$-divisor is relatively ample iff it is a positive linear combination of ample $\mathbb{Q}$-divisors.

Note that if U is projective, then H is relatively ample iff there is an ample divisor G on U, such that $H + \pi^* G$ is ample. Note also that an $\mathbb{R}$-divisor is relatively ample iff defines a positive linear functionall on $\overline{\mathrm{NE}}(X/U) - \{0\}$. Note that many of the definitions for $\mathbb{Q}$-divisors extend to $\mathbb{Q}$-divisors. In particular, the property of being nef and the numerical dimension.

Proof of (7.3). Pick an extremal ray $R = \mathbb{R}^+ \alpha$ of the closed cone of curves. We may pick an ample $\mathbb{R}$-divisor H such that

$$(K_S + H) \cdot \beta \geq 0,$$

for all $\beta \in \overline{\mathrm{NE}}(S)$ with equality iff $R = \mathbb{R}^+ \beta$. In particular $D = K_S + H$ is a nef $\mathbb{R}$-Cartier divisor. The key technical point is to establish that R is rational, so that we may choose H to be an ample $\mathbb{Q}$-divisor. In fact we will prove much more, we will prove that R is spanned by a curve. Let $\nu = \nu(S, D)$. There are three cases:

- $\nu = 0$,
- $\nu = 1$, and
- $\nu = 2$.

If $\nu = 0$, then $K_S + H$ is numerically trivial, and $-K_S$ is numerically equivalent to H. In other words $-K_S$ is ample. Moreover every curve C spans R. Thus S is a Fano surface of Picard number one and it follows that $S \simeq \mathbb{P}^2$. Note that R is rational in this case.

If $\nu = 1$, then we will defer the proof that R is rational. So assume that H is rational. We first prove that D is semiample. Consider asymptotic Riemann-Roch. $D^2 = 0$, by assumption.

$$D \cdot (-K_S) = D \cdot H > 0,$$

also by assumption. Thus $\chi(X, \mathcal{O}_X(mD))$ grows linearly. Since

$$h^2(S, \mathcal{O}_S(mD)) = h^0(S, \mathcal{O}_S(K_S - mD)) = 0,$$

for m sufficiently large, it follows that there is a positive integer $m > 0$ such that $|mD|$ contains a pencil. By (7.10) it follows that D is semi-ample. Let F by the general fibre of the corresponding morphism $\pi\colon S \longrightarrow C$. Then F is a smooth irreducible curve, $F^2 = 0$ and $-K_S \cdot F > 0$. By adjunction,

$$0 > (K_S + F) \cdot F = K_F = 2g - 2.$$

It follows that $g = 0$ so that $F \simeq \mathbb{P}^1$. Moreover since $R = \mathbb{R}^+[F]$ is extremal, the relative Picard number is one and so there are no reducible fibres. By direct classification it follows that there are no singular fibres. Thus π is a $\mathbb{P}^1$-bundle.

If $\nu = 2$ then D is big but not ample. As D is nef $D^2 > 0$. By continuity there is an ample $\mathbb{Q}$-divisor G such that $(K_S + G)^2 > 0$ and $(K_S + G) \cdot G > 0$, where $H - G$ is ample. Thus $K_S + G$ is big. By Kodaira's Lemma, $K_S + G \sim_{\mathbb{Q}} A + E$, where A is ample and $E \geq 0$. Now

$$(K_S + G) \cdot \alpha = D \cdot \alpha - (H - G) \cdot \alpha < 0.$$

Since $\alpha \in \overline{\mathrm{NE}}(S)$ we may find $\alpha_i = \lambda_i[C_i]$, where $\lambda_i > 0$ are positive rational numbers and C_i are curves. But then

$$0 > D \cdot C_i = A \cdot C_i + E \cdot C_i > E \cdot C_i,$$

for i sufficiently large. But then C_i is a component of E and C_i has negative self-intersection. As E has only finitely many components, it follows that $R = \mathbb{R}^+[C]$, for some $C = C_i$. In particular R is rational and we choose H to be a $\mathbb{Q}$-divisor. We have

$$0 > (K_S + C) \cdot C = K_C = 2g - 2.$$

Thus $g = 0$ and C is a -1-curve.

Replacing H by a multiple, we may assume that H is very ample (for the time being, we only need that is Cartier). Suppose that $H \cdot C = m > 0$. We may always assume that $m > 1$ (simply replace H by a multiple). If $G = H + (m-2)C$, then G is big and it is nef, since $G \cdot C = 2$. In particular G is ample by Nakai-Moishezon. Let $D = K_S + C + G$. Since we may write

$$D = (K_S + H) + C,$$

the stable base locus of D is contained in C. In particular for every curve $\Sigma \subset S$,

$$D \cdot \Sigma \geq 0,$$

with equality iff $\Sigma = C$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(D - C) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0.$$

Now $\mathcal{O}_C(D) = \mathcal{O}_C$, since $D|_C$ is a divisor of degree zero on $\mathbb{P}^1$. On the other hand,

$$H^1(S, \mathcal{O}_S(D - C)) = H^1(S, \mathcal{O}_S(K_S + G)) = 0,$$

by Kodaira vanishing. Thus there are no base points of D on C, so that D is semiample, and the resulting morphism $\pi\colon S \longrightarrow Z$ contracts C.

It remains to prove that Z is smooth. Consider the ample divisor $K_S + G$. We may always pick H very ample. In this case, I claim that $K_S + G$ is base point free (in fact it is very ample). The base locus is supported on C. Consider the exact

$$0 \longrightarrow \mathcal{O}_S(K_S + G - C) \longrightarrow \mathcal{O}_S(K_S + G) \longrightarrow \mathcal{O}_C(K_S + G) \longrightarrow 0.$$

Then $\mathcal{O}_C(K_S + G) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. As before,

$$H^1(S, \mathcal{O}_S(K_S + H + (m-3)C) = 0,$$

by Kodaira vanishing. Thus $K_S + G$ is base point free. Pick a general curve $\Sigma' \in |K_S + G|$. Then this must intersect C transversally at a single smooth point. But then $\Sigma = \pi_*\Sigma'$ is a smooth curve in Z. On the other hand $\Sigma + C \in |D|$. Since $|D|$ defines the contraction, the image of $\Sigma + C$, which is again Σ' is Cartier.

But any variety which contains a smooth Cartier divisor, is smooth in a neighbourhood of the divisor. Thus Z is smooth. $\square$

(7.3) allows us to define the K_S-MMP for surfaces. The aim of the minimal model program is to try to make K_S nef.

(1) Start with a smooth projective surface S.
(2) Is K_S nef? Is yes, then stop.
(3) Otherwise there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(S)$ on which K_S is negative. By (7.3) there is a contraction $\pi\colon S \longrightarrow Z$ of R.
   - **Mori fibre space:** If $\dim Z \leq 1$ then the fibres of π are Fano varieties.
   - **Birational contraction:** In this case replace S by Y and return to (2).

In other words, the K_S-MMP produces a sequence of smooth surface $\pi_i\colon S_{i-1} \longrightarrow S_i$, where each π_i blows down a -1-curve (conversely each π_i blows up a smooth point of S_i), starting with $S_0 = S$. This process must terminate, since the Picard number of S_i is one less than the Picard number of S_{i-1}. At the end we have a smooth surface $T = S_k$, such that either K_T is nef or $\pi\colon T \longrightarrow C$ is $\mathbb{P}^1$-bundle over a curve, or $T \simeq \mathbb{P}^2$.

Definition 7.12. *Let X be a normal variety.*

We say that X is a ***Fano variety*** *if X is projective and $-K_X$ is ample.*

We say that a projective morphism $\pi\colon X \longrightarrow Z$ is a ***Fano fibration*** *if $-K_X$ is π-ample.*

*Let R be an extremal ray of the closed cone of curves of X. We say that R is K_X-**extremal** if $K_X \cdot R < 0$. We say $\pi\colon X \longrightarrow Z$ is the* ***contraction associated to*** *R if π is a contraction morphism and C is contracted iff $R = \mathbb{R}^+ C$.*

Lemma 7.13. *Let S be a smooth surface. Then the log discrepancy of S is equal to 2 and the only valuations of log discrepancy 2 are given by blowing up a point.*

Proof. Easy calculation. □

Proof of (7.5)*.* One direction is clear. If $\pi\colon S \longrightarrow T$ blows up p, and C is the exceptional divisor, then we have already seen that $C^2 = -1$, and $C \simeq \mathbb{P}^1$. But then by adjunction

$$2g - 2 = -2 = K_C = (K_S + C) \cdot C = K_S \cdot C - 1.$$

Thus $K_S \cdot C = -1$ and C is a -1-curve.

Now suppose that C is a -1-curve. Then $R = \mathbb{R}^+[C]$ is a K_S-extremal ray of the cone of curves. Let $\pi\colon S \longrightarrow T$ be the associated contraction morphism. We will prove later that T is smooth (this is easy, when viewed the right way). Now suppose that we write

$$K_S + C = \pi^* K_T + aC.$$

Dotting both sides by C, we see that $a = 2$, and we are done by (7.13). □

The MMP for surfaces can be extended in two interesting (but essentially trivial) ways. The first way we restrict the choice of extremal rays to contract and the second way we group together extremal rays and contract them simultaneously.

First suppose we are given a projective morphism $g\colon S \longrightarrow U$. Then one can ensure that every step of the MMP lies over U, simply by only contracting rays of the relative cone of curves, $\overline{\mathrm{NE}}(S/U)$. At the end, either K_S is nef over U (meaning that it is nef on every curve contracted over U) or we get a Mori fibre space over U.

Secondly suppose we have a group G acting on S. By simultaneously contracting whole faces of the cone of curves, which are orbits of a single extremal ray, the resulting contraction is then G-equivariant. This gives us a K_S-MMP which preserves the action of G. Note though, that the relative Picard number of each step can be larger than one (in fact the relative Picard number of the G-invariant part is always one).

A particularly interesting case, is when S is a smooth surface defined over the real numbers. In this case, we let G be the Galois group of $\mathbb{C}$ over $\mathbb{R}$ (namely $\mathbb{Z}_2$, generated by complex conjugation). The resulting steps of the MMP respect the action of complex conjugation, so that the MMP is defined over $\mathbb{R}$. Clearly similar remarks hold for other non-algebraically closed fields.

Theorem 7.14 (Hodge Index Theorem)**.** *Let S be a smooth projective surface.*

Then the intersection pairing

$$NS(S) \times NS(S) \longrightarrow \mathbb{R},$$

has signature $(+, -, -, -, \dots, -)$.

In particular if $D^2 > 0$ and $D \cdot E = 0$ then $E^2 \leq 0$ with equality iff E is numerically trivial.

Proof. It suffices to prove the last statement for any D such that $D^2 > 0$. So we may assume that D is ample. Suppose that $E^2 \geq 0$. Consider $H = D + mE$, where m is large. Then H is ample by Nakai-Moishezon.

Suppose that $E^2 > 0$. As $H \cdot E > 0$, it follows that $\kappa(S, E) > 0$, by Asymptotic Riemann-Roch. But then $D \cdot E > 0$, a contradiction.

Now suppose that $E^2 = 0$ but that E is not numerically trivial. Then there is a curve C such that $E \cdot C \neq 0$. Let $E' = (D \cdot C)D - (D^2)C$. Then $E' \cdot D = 0$, but $E' \cdot C \neq 0$. Thus replacing E by $E \pm E'$ we are reduced to the case when $E^2 > 0$. □

Lemma 7.15 (Negativity of Contraction)**.** *Let $\pi \colon X \longrightarrow U$ be a proper birational morphism of varieties and let B be a $\mathbb{Q}$-Cartier divisor.*

If $-B$ is π-nef then $B \geq 0$ iff $\pi_* B \geq 0$.

Proof. One direction is clear, if $B \geq 0$ then $\pi_* B \geq 0$.

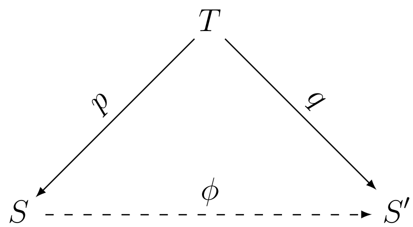
Otherwise, we may assume that X and U are normal, and U is affine. Cutting by hyperplanes, we may assume that U is a surface. Passing to a resolution of X, we may assume that X is a smooth surface. Compactifying X and U we may assume that X and U are projective. Let $D = \pi^* H$ and list the exceptional divisors $E_1, E_2, \dots, E_k$. Then $D^2 > 0$ and $D \cdot E_i = 0$. It follows that the intersection matrix $(E_i \cdot E_j)$ is negative definite. Suppose that $B = \sum b_i E_i + B'$, where no component of $B' \geq 0$ is exceptional. Then

$$(\sum b_i E_i) \cdot E_j \leq B \cdot E_j < 0.$$

Thus $b_i \geq 0$. □

Theorem 7.16 (Strong Factorisation)**.** *Let $\phi\colon S \dashrightarrow S'$ be a birational map between two smooth projective surfaces.*

Then there are two birational maps $p\colon T \longrightarrow S$ and $q\colon T \longrightarrow S'$ which are both compositions of smooth blow ups of smooth points (and isomorphisms) and a commutative diagram



Proof. By elimination of indeterminancy we may assume that q is a composition of smooth blow ups. Replacing ϕ by p, we may therefore assume that $\phi\colon T \longrightarrow S$ is a birational morphism.

Consider running the K_T-MMP over S. This terminates with a relative minimal model, $\pi\colon T \longrightarrow T'$ over S. The morphism π contracts -1-curves, and so π is a composition of smooth blow ups. It suffices to show that $T' = S$. Replacing T by T', it suffices to show that if K_T is nef then $T = S$.

Suppose not. We may write

$$K_T + E = \pi^* K_S + \sum a_i E_i.$$

Since S is smooth it has log discrepancy two and so each $a_i \geq 2$. But then if we write

$$K_T = \pi^* K_S + B,$$

then $B \geq 0$ contains the full exceptional locus. By negativity of contraction, there is an exceptional divisor E_i such that $B \cdot E_i < 0$. But then K_T is not nef, a contradiction. □

Lemma 7.17. *Let (X, Δ) be a log pair.*

If X is a curve and $K_X + \Delta$ is nef then it is semiample.

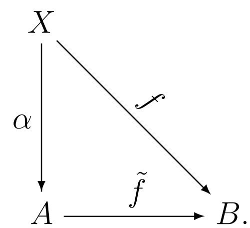
Proof. Let ν be the numerical dimension of $K_X + \Delta$, and let d be the degree. There are two cases:

(1) $d = 0$ and $\nu = 0$.
(2) $d > 0$ and $\nu = 1$.

If $d > 0$ then $K_X + \Delta$ is ample and there is nothing to prove. If $d = 0$ there are two cases. If $g = 1$ then Δ is empty and $K_X \sim 0$ as X is an elliptic curve. If $g = 0$ then $X \simeq \mathbb{P}^1$. Pick $m > 0$ such that $D = m(K_X + \Delta)$ is integral. Then $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}$, so that $|D|$ is base point free. □

Definition 7.18. *Let X be a smooth projective variety. Then there is a morphism $\alpha\colon X \longrightarrow A$ to an Abelian variety, which is universal amongst all such morphisms in the following sense:*

Let $f\colon X \longrightarrow B$ be another morphism to an a abelian variety. Then there is a morphism $\tilde{f}\colon A \longrightarrow B$ and a commutative diagram



In characteristic zero, α induces an isomorphism

$$H^1(X, \mathcal{O}_X) \simeq H^1(A, \mathcal{O}_A).$$

In particular $\dim A = q(X)$.

Lemma 7.19 (Kodaira's Formula)**.** *Let $\pi\colon S \longrightarrow C$ be a contraction morphism, where S is a smooth projective surface and C is a smooth projective curve and the generic fibre is an elliptic curve.*

If K_S is nef over C then there is a divisor $\Delta \geq 0$ on C such that

$$K_S = \pi^*(K_C + \Delta).$$

Sketch of proof of (7.6). Let $\nu = \nu(S, K_S)$ be the numerical dimension. There are three cases.

If $\nu = 0$ then K_S is numerically trivial. Let $\alpha\colon S \longrightarrow A$ be the Albanese morphism. Let Z be the image. There are three cases, given by the dimension of Z.

If $q = 0$, equivalently Z is a point, then every numerically trivial divisor is torsion and there is nothing to prove. Suppose that $Z = C$ is a curve. Let F be a general fibre. Then

$$2g - 2 = K_F = (K_S + F) \cdot F = 0,$$

so that $g = 1$ and F is an elliptic curve. But then the result follows by (7.19) and (7.17). Thus we may assume that Z is either a surface or a point. Finally suppose that Z is a surface. With some work, one shows that $Z = A$, and that α is birational, whence an isomorphism.

If $\nu = 1$ we first assume that $q = 0$. But then,

$$\chi(S, \mathcal{O}_S) = 1 - q + p_g > 0.$$

Riemman-Roch then reads

$$h^0(S, \mathcal{O}_S(mK_S)) \geq \chi(S, \mathcal{O}_S(mK_S)) = \chi(S, \mathcal{O}_S)) > 0.$$

It follows that $|mK_S| \neq \varnothing$, for $m > 0$. Let $C \in |mK_S|$. Then

$$2g - 2 = K_C = (K_S + C) \cdot C = (m+1)C|_C = 0.$$

But then C is a smooth curve of genus one (or a rational curve with a single node or cusp). Moreover since $K_C \sim 0$ it follows that $C|_C$ is torsion. Now there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_S((k-1)C) \longrightarrow \mathcal{O}_S(kC) \longrightarrow \mathcal{O}_C(kC) \longrightarrow 0.$$

Note that $\mathcal{O}_C(kC) = \mathcal{O}_C$ infinitely often. Therefore $h^1(C, \mathcal{O}_C(kC)) \neq 0$ infinitely often. It follows that $h^1(S, \mathcal{O}_S(kC))$ is an unbounded function of k. Since

$$\chi(S, \mathcal{O}_S(kC)) \geq 0 \qquad \text{and} \qquad h^2(S, \mathcal{O}_S(kC)) = 0,$$

for $k \geq 2$, it follows that $h^0(S, \mathcal{O}_S(kC))$ is an unbounded function of k and we are done by (7.10). If $\nu = 1$ and $q > 0$, then we again have to carefully analyse the map α.

If $\nu = 2$ then $K_S^2 > 0$. If K_S is ample there is nothing to prove. Otherwise, by Nakai-Moishezon, there is a curve C such that $K_S \cdot C = 0$. By Kodaira's Lemma, $K_S \sim A + E$, where A is ample and E is effective. But then C is a component of E and C has negative self-intersection. We have

$$2g - 2 = K_C = (K_S + C) \cdot C < 0.$$

But then $g = 0$, $C \simeq \mathbb{P}^1$, and $C^2 = -2$ (C is then called a -2-curve).

With some work, we can contract C, as before, $\pi \colon S \longrightarrow T$. In fact $K_S = \pi^* K_T$. Repeating this process, as in the K_S-MMP, we reduce to the case when K_S is ample. The only twist is that if we contract any curves, the resulting surface is necessarily singular. □

Proof of (7.7). Modulo some interesting details which we will skip, this essentially follows by applying the K_S-MMP and considering the maps given by abundance and the Albanese. □