4. Some examples

We present some examples of varieties, mainly surfaces, with interesting Mori cones.

Definition 4.1. *Let $\mathcal{C} \subset V \simeq \mathbb{R}^n$ be a subset of a finite dimensional real vector space. We say that $\mathcal{C}$ is a* ***cone*** *(respectively* ***convex subset****) if whenever α and $\beta \in \mathcal{C}$ then*

$$\lambda\alpha + \mu\beta \in \mathcal{C} \qquad \text{for all} \qquad \lambda \geq 0, \mu \geq 0,$$

(respectively such that $\lambda + \mu = 1$). We say that $\mathcal{C}$ is ***strictly convex*** *if $\mathcal{C}$ contains no positive dimensional linear subspaces.*

We say that $R \subset \mathcal{C}$ is a ***ray*** *of a cone $\mathcal{C}$ if $R = \mathbb{R}^+\alpha$, for some non-zero vector $\alpha \in \mathcal{C}$. We say that R is an* ***extremal ray*** *if whenever $\beta + \gamma \in R$, where β and $\gamma \in \mathcal{C}$, then β and $\gamma \in R$.*

It follows by Kleiman's criteria that the closed cone of curves of a projective variety is strictly convex.

Lemma 4.2. *Let S be a smooth projective surface and let $\alpha = \sum a_i[C_i]$ and $\beta = \sum b_j[C_j]$ be two cycles, where $a_i > 0$ and $b_j > 0$.*

If $\alpha \cdot \beta < 0$ then $C = C_i = C_j$ for some i and j, where $C^2 < 0$. If i and j are the only two indices with this property then $a_ib_jC^2 \leq \alpha \cdot \beta$.

Proof. Clear, since

$$\alpha \cdot \beta = \sum a_ib_jC_i \cdot C_j,$$

and $C_i \cdot C_j \geq 0$ unless $C_i = C_j$. □

Lemma 4.3. *Let S be a smooth projective surface. Let*

1. *If R is $R = \mathbb{R}^+\alpha \subset \overline{\mathrm{NE}}(S)$ is an extremal ray of the closed cone of curves of S then $\alpha^2 \leq 0$.*
2. *If C is an irreducible curve such that $C^2 < 0$ then $R = \mathbb{R}^+[C]$ is extremal.*

Proof. We will first show that $R = \mathbb{R}^+\alpha$ is never extremal if $\alpha^2 > 0$. Let H be an ample divisor. Then $H \cdot \alpha > 0$ by Kleiman's criteria. Pick a small neighbourhood U of $\alpha \in V$ such that

- $\beta^2 > 0$
- $\beta \cdot H > 0$,

for all $\beta \in U$. Suppose that $\beta \in U$ is rational. Pick $k \in \mathbb{N}$ such that $D = k\beta$ is integral. By Asymptotic Riemann-Roch,

$$\begin{aligned} h^0(S, \mathcal{O}_S(mD)) + h^0(S, \mathcal{O}_S(K_S - mD)) &= h^0(S, \mathcal{O}_S(mD)) + h^2(S, \mathcal{O}_S(mD)) \\ &\geq \chi(S, \mathcal{O}_S(mD)) > 0, \end{aligned}$$

for large m. Thus either $|mD|$ or $|K_S - mD|$ is non-empty for large m. But $(K_S - mD) \cdot H < 0$ so that $|(K_S - mD)|$ is empty and so $|mD|$ is non-empty. Thus $[D] \in \mathrm{NE}(S)$ and so $U \subset \overline{\mathrm{NE}}(S)$. In particular R is not extremal.

Now suppose that C is an irreducible curve such that $C^2 \leq 0$. Then we may write

$$[C] = \sum \beta_i$$

where the β_i generate extremal rays of $\overline{\mathrm{NE}}(S)$. As

$$0 > C^2 = [C] \cdot (\sum \beta_i).$$

it follows that $[C] \cdot \beta_i < 0$ for some i. Since β_i is a limit of $\beta(j) \in \mathrm{NE}(S)$, (4.2) implies that $\beta_i = \lambda[C] + \beta'$, where $\beta' \in \overline{\mathrm{NE}}(S)$ and $\lambda > 0$. As β_i generates an extremal ray, it follows that $\mathbb{R}^+\beta_i = \mathbb{R}^+[C] = R$ is extremal. $\square$

Let $\pi \colon X \longrightarrow C$ be a $\mathbb{P}^r$-bundle over a smooth curve. We recall the classification of such bundles. We have that $S = \mathbb{P}(E)$, for some rank $r+1$ vector bundle over C and the two $\mathbb{P}^r$-bundles $S_i = \mathbb{P}(E_i)$ are isomorphic over C iff there is a line bundle L and an isomorphism of vector bundles $E_1 \otimes L \simeq E_2$.

Now the Picard group of X has rank two, and it is generated by the class of a line in a fibre and the class of any section. Thus $\overline{\mathrm{NE}}(X) \subset \mathbb{R}^2$. Taking a compact slice, we get a closed interval, so that topologically the situation is an open and closed book. To get a complete description, we have two rays R_1 and R_2 and it suffices to determine generators for each ray.

First suppose that $C = \mathbb{P}^1$, so that $X = S$ is a rational surface, a $\mathbb{P}^1$-bundle over C. In this case any rank two vector bundle E has the form $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ by a Theorem of Grothendieck. We may normalise so that $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. The resulting surface is denoted $\mathbb{F}_n$. Let f denote the class of a fibre and e the class of a section of minimal self-intersection $-n$. Then

$$\overline{\mathrm{NE}}(X) = \mathbb{R}^+ f + \mathbb{R}^+ e.$$

But now suppose that C has higher genus. We need to say something about all sections and multisections of π. Fortunately in characteristic zero we only need to keep track of the multisections. We recall some of the theory of vector bundles:

Definition 4.4. *Let* $f\colon E \longrightarrow C$ *be a vector bundle over a curve. The* ***slope*** *of* E *is the rational number*

$$\mu(E) = \frac{\deg E}{\operatorname{rk} E}.$$

We say that E *is* ***stable*** *(respectively* ***semi-stable****) if for all quotient vector bundles* $E \longrightarrow F$ *of* E*, we have*

$$\mu(E) < \mu(F) \qquad \text{(respectively} \qquad \mu(E) \leq \mu(F)).$$

We say that E *is* ***unstable*** *if it is not semistable. We say that* F ***destabilises*** E *if* F *is a quotient* $E \longrightarrow F$ *of* E *and*

$$\mu(F) < \mu(E).$$

The maximal ***destabilising quotient*** *is a quotient vector bundle with the smallest slope and the largest rank amongst quotients with the same slope.*

Example 4.5. *Suppose that*

$$E = \bigoplus_{i=0}^{r} L_i,$$

is a direct sum of line bundles. If $\deg L_i = d_i$*, then*

$$\mu(E) = \frac{d_0 + d_1 + \cdots + d_r}{r+1} \qquad \text{and} \qquad \mu(L_i) = d_i.$$

As $F = L_i$ *is a quotient of* E*,* E *is never stable and it is semistable iff* $\deg L_i$ *is independent of* i*. Let* $m = \min_i d_i$*. If* $m \neq \mu(E)$*, then the maximal destabilising quotient is*

$$\bigoplus_{i:d_i=m}^{r} L_i.$$

Lemma 4.6. *Let* E *be a vector bundle over a smooth curve* C*.*

Then E *is semi-stable iff* $E' = f^*E$ *is semi-stable for all covers* $f\colon C' \longrightarrow C$*.*

Proof. One direction is clear; if $E \longrightarrow F$ destabilises E then $E' \longrightarrow F' = f^*F$ destabilises E'.

Suppose that E' is not semi-stable. By what we already proved, passing to a finite cover of C' we may assume that f is Galois, with Galois group $G \subset \operatorname{Aut}(C')$. Let $E' \longrightarrow F'$ be a maximal destabilising subsheaf. Then F' is canonical, whence invariant under the action of $G \subset \operatorname{Aut}(C')$. But this means that $F' = f^*F$ for some vector bundle F and F destabilises E. □

Remark 4.7. *It is not true that if E is stable then f^*E is stable. It can happen that f^*E is semistable. Also it is not true that if V is an arbitrary vector bundle on C' which is invariant under G then $V = f^*W$, for some vector bundle W on C. In fact this can fail even for line bundles. We need the fact F' is a quotient of $E' = f^*E$.*

Lemma 4.8. *Let C be a smooth curve and let $\pi\colon X = \mathbb{P}(E) \longrightarrow C$ be a $\mathbb{P}^r$-bundle over C. Suppose that $r > 1$.*

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(1) *E is stable (respectively semi-stable).*
(2) *$-K_{X/C} \cdot \Sigma > 0$ (respectively ≥ 0) for all curves $\Sigma \subset X$, where $K_{X/C} = K_X - \pi^*K_C$ is the relative canonical divisor.*

Example 4.9. *Let C be a curve of genus at least two. Suppose that E is a general rank two stable vector bundle, which admits an exact sequence*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where L is a line bundle of positive degree $d \leq g$ (such exist by general theory). If E chosen generically (see for example [1]). Let $S = \mathbb{P}(E)$. Then there is a section C_0 of minimal self-intersection d.

Suppose that Σ is a multi-section of S, so that $f\colon D = \Sigma \longrightarrow C$ is dominant. I claim that $\Sigma^2 > 0$. Let $Y = \mathbb{P}(f^*E)$. Then

$$f^*\Sigma = \Sigma_0 + \Sigma_1,$$

where Σ_0 is a section of $Y \longrightarrow D$. But then

$$\Sigma^2 = \Sigma^0 \cdot f^*\Sigma \geq \Sigma_0^2.$$

As f^*E is stable we may replace X by Y and Σ by Σ_0^2, so that we may assume that Σ is a section of $X \longrightarrow C$. But then

$$(K_X + \Sigma) \cdot \Sigma = K_\Sigma = \pi^*K_C.$$

Thus

$$\Sigma^2 = -K_{X/C} \cdot \Sigma > 0,$$

by assumption. Consider $\overline{\mathrm{NE}}(S)$. One edge is given by f the class of a fibre. What about the other edge? Suppose that this is generated by α. All curves other than F have positive self-intersection. By (4.3) we must have $\alpha^2 = 0$. Thus $\mathrm{NE}(S)$ is not a closed cone.

Rescaling we may suppose that $\alpha = \sigma - af$ σ is the class of Σ and $a \geq 0$. We have

$$d - 2a = (\alpha)^2 = 0.$$

Thus the divisor $D = 2\Sigma - dF = -K_{X/C}$ is a divisor which intersects every curve positively, but which is not ample. This gives an example in characteristic zero where Kleiman's criteria is sharp.

Example 4.10. *Let $S = E \times E$ the product of two general elliptic curves. Then $\rho(S) = 3$. Let f_i be the class of a fibre and let δ be the class of the diagonal. Suppose that $\delta = a_1 f_1 + a_2 f_2$. Then*

$$a_i = (a_1 f_1 + a_2 f_2) \cdot a_{2-i} = \delta \cdot f_{2-i} = 1.$$

But

$$0 = \delta^2 \neq (f_1 + f_2)^2 = 2.$$

Thus f_1 and f_2 and δ define independent classes, which actually span the Néron-Severi group.

On the other hand let $D \geq 0$ be a $\mathbb{Q}$-Cartier divisor. Then $D^2 \geq 0$ with equality iff $D = 0$, as can be seen by acting by a general translation. Thus $\overline{\mathrm{NE}}(S)$ is half of a classical cone. There are uncountably many extremal rays, and at most countably many contractions. Most rays are not rational.

Example 4.11. *Let S be obtained from $\mathbb{P}^2$ by blowing up nine points $p_1, p_2, \dots, p_9$. Suppose first that these points are the nine points of the intersection of two general smooth cubics. Then S is the total space of the pencil, and there is a morphism $f \colon S \longrightarrow \mathbb{P}^1$ whose fibres are the elements of the pencil. The nine exceptional divisors $E_1, E_2, \dots, E_9$ are then sections of this fibration. The generic fibre C of f is an elliptic curve (over the function field $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$) and the nine sections define nine points $e_1, e_2, \dots, e_9$. Since the pencil is general, it follows that these points generate a subgroup of C, isomorphic to $\mathbb{Z}^8$. This subgroup then corresponds to a subgroup of the automorphism group of S over the base. The orbit of the nine exceptional divisors, gives infinitely many exceptional divisors. Each exceptional divisor generates an extremal ray of the closed cone of curves.*

What is worse, -1-curves persist under small deformations. If we therefore perturb these nine points to nine very general points, infinitely many of these -1-curves survive. The resulting surface does not have any automorphisms, and yet its Mori cone is still very complicated.

Let us end these series of examples with Zariski's famous example:

Example 4.12. *Pick a smooth cubic curve C in $\mathbb{P}^2$ and let S be the blow up of $f \colon S \longrightarrow \mathbb{P}^2$ at ten very general points $p_1, p_2, \dots, p_{10}$ of C. Let $E_1, E_2, \dots, E_{10}$ be the ten exceptional divisors and let Σ be the strict transform of C.*

Then Σ is a curve of self-intersection $9 - 10 = -1$. By a result due to Artin, there is a contraction morphism $\pi \colon S \longrightarrow T$ contracting Σ, where T is a normal algebraic space (or if you will an analytic space). I claim that T is not a projective variety.

Suppose it were. Then T would have an ample divisor D. But I claim it has no Cartier divisors at all. We have

$$\pi^* D \in A_1(S) = \mathbb{Z}^{10} = \mathbb{Z}[f^* H] + \sum_{i=1}^{9} \mathbb{Z}[E_i],$$

where H is a line in $\mathbb{P}^2$. So

$$\pi^* D \sim a\pi^* H - \sum a_i E_i.$$

Now

$$\pi^* D \cdot \Sigma = D \cdot \pi_* \Sigma = 0.$$

Thus

$$3a = \sum a_i.$$

Moreover $\pi^* D|_\Sigma$ would be linearly equivalent to zero. Thus there would be some curve B of degree d in $\mathbb{P}^2$ such that

$$B|_C \sim \sum b_i p_i.$$

But this contradicts the fact that our ten points of C are general.

Note that there are then plenty of nef divisors D on S which are zero on C but which are not semiample (since if D were semiample, it would descend to T).

Definition 4.13. *A **stable n-pointed curve of genus** g is a connected curve of arithmetic genus g, with only nodes as singularities, with n marked points contained in the smooth locus, such that the normalisation of every component isomorphic to $\mathbb{P}^1$ has three special points (either a node or a marked point).*

The moduli space of genus g, n-pointed stable curves $\overline{M}_{g,n}$ is a projective variety whose points are in natural correspondence with isomorphism classes of stable n-pointed curves of genus g. Note that $\overline{M}_{1,1} \simeq \overline{M}_{0,4} \simeq \mathbb{P}^1$. Indeed, the second isomorphism is given by the j-invariant, and the first is a consequence since an elliptic curve double covers $\mathbb{P}^1$ over 4 points.

These gives finitely many rational curves in $\overline{M}_{g,n}$, which we call vital curves. Indeed take a stable curve of genus $g-1$ with n points (respectively a stable curve of genus g with $n-3$ points) whose components are copies of $\mathbb{P}^1$ with 3 labelled points, together with one component with only two labelled points. Now attach an elliptic to the special component (or a copy of $\mathbb{P}^1$ with three marked points). The resulting curve is a point of $\overline{M}_{g,n}$. Varying the moduli of the elliptic curve (or of the four points), gives a curve $C \subset \overline{M}_{g,n}$.

Conjecture 4.14 (Faber, Fulton, Mumford). *$\overline{M}_{g,n}$ is spanned by the classes of the vital curves.*

Theorem 4.15 (Keel, Gibney, Morrison). *To prove* (4.14) *it suffices to prove the case when $g = 0$.*

References

[1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin, 1970.