Birational boundedness

James M^{c}Kernan

MIT

Birational automorphisms

heorem: (Hacon, -, Xu) If X is a smooth projective variety of general type, then the number of birational automorphisms of X is bounded by $c \cdot \mathrm{vol}(X, K_X)$, where c is a constant depending only on the dimension of X.

Birational automorphisms

heorem: (Hacon, -, Xu) If X is a smooth projective variety of general type, then the number of birational automorphisms of X is bounded by $c \cdot \mathrm{vol}(X, K_X)$, where c is a constant depending only on the dimension of X.

finition: The volume of a divisor D is

$$\mathrm{vol}(X, D) = \limsup \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n},$$

where n is the dimension of X.

Birational automorphisms

heorem: (Hacon, -, Xu) If X is a smooth projective variety of general type, then the number of birational automorphisms of X is bounded by $c \cdot \mathrm{vol}(X, K_X)$, where c is a constant depending only on the dimension of X.

finition: The volume of a divisor D is

$$\mathrm{vol}(X, D) = \limsup \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n},$$

where n is the dimension of X.

- When D is nef this is nothing more than the degree D^n of D, by asymptotic Riemann-Roch.

Optimal value for c?

- A smooth curve C is of general type if and only if the genus $g \geq 2$; the volume of K_C is then $2g - 2$.

Optimal value for c?

- A smooth curve C is of general type if and only if the genus $g \geq 2$; the volume of K_C is then $2g - 2$.
- The size of the automorphism group of C is at most $84(g-1) = 42(2g-2)$, so we may take $c = 42$.

Optimal value for c?

- A smooth curve C is of general type if and only if the genus $g \geq 2$; the volume of K_C is then $2g - 2$.
- The size of the automorphism group of C is at most $84(g-1) = 42(2g-2)$, so we may take $c = 42$.
- There are infinitely many values of g for which there is (or is not) a smooth curve of genus g with $84(g-1)$ automorphisms.

Optimal value for c?

- A smooth curve C is of general type if and only if the genus $g \geq 2$; the volume of K_C is then $2g-2$.
- The size of the automorphism group of C is at most $84(g-1) = 42(2g-2)$, so we may take $c = 42$.
- There are infinitely many values of g for which there is (or is not) a smooth curve of genus g with $84(g-1)$ automorphisms.

heorem: (Xiao) If S is a smooth projective surface of general type, then the size of the automorphism group is at most $(42)^2 \operatorname{vol}(S, K_S)$, with equality if and only if $S = C \times C$ and C has $84(g-1)$ automorphisms.

Optimal value for c?

- A smooth curve C is of general type if and only if the genus $g \geq 2$; the volume of K_C is then $2g - 2$.
- The size of the automorphism group of C is at most $84(g-1) = 42(2g-2)$, so we may take $c = 42$.
- There are infinitely many values of g for which there is (or is not) a smooth curve of genus g with $84(g-1)$ automorphisms.

heorem: (Xiao) If S is a smooth projective surface of general type, then the size of the automorphism group is at most $(42)^2 \operatorname{vol}(S, K_S)$, with equality if and only if $S = C \times C$ and C has $84(g-1)$ automorphisms.

uestion: Is $c = (42)^n$ the optimal constant in dimension n?

Canonical series

- The birational automorphism group acts on the linear series $|mK_X|$.

Canonical series

- The birational automorphism group acts on the linear series $|mK_X|$.
- The automorphism group of a curve permutes the Weierstrass points, the inflection points of the canonical curve $C \subset \mathbb{P}^{g-1}$.

Canonical series

- The birational automorphism group acts on the linear series $|mK_X|$.
- The automorphism group of a curve permutes the Weierstrass points, the inflection points of the canonical curve $C \subset \mathbb{P}^{g-1}$.
- There are $g^3 - g$ Weierstrass points, up to multiplicity and this gives a lot of control.

Canonical series

- The birational automorphism group acts on the linear series $|mK_X|$.
- The automorphism group of a curve permutes the Weierstrass points, the inflection points of the canonical curve $C \subset \mathbb{P}^{g-1}$.
- There are $g^3 - g$ Weierstrass points, up to multiplicity and this gives a lot of control.
- This approach does not seem to generalise well to higher dimensions; it is hard to generalise to higher dimensions the notion of a Weierstrass point.

Canonical series

- The birational automorphism group acts on the linear series $|mK_X|$.
- The automorphism group of a curve permutes the Weierstrass points, the inflection points of the canonical curve $C \subset \mathbb{P}^{g-1}$.
- There are $g^3 - g$ Weierstrass points, up to multiplicity and this gives a lot of control.
- This approach does not seem to generalise well to higher dimensions; it is hard to generalise to higher dimensions the notion of a Weierstrass point.
- Instead we merge the approaches of Alexeev and Tsuji.

Riemann-Hurwitz

- Let C be a smooth curve of genus g with automorphism group G and let $\pi\colon C \longrightarrow B = C/G$ be the quotient morphism.

Riemann-Hurwitz

- Let C be a smooth curve of genus g with automorphism group G and let $\pi\colon C \longrightarrow B = C/G$ be the quotient morphism.
- Then $K_C = \pi^*(K_B + \Delta)$, where $\Delta = \sum \frac{r_i - 1}{r_i} p_i$ comes from Riemann-Hurwitz.

Riemann-Hurwitz

- Let C be a smooth curve of genus g with automorphism group G and let $\pi\colon C \longrightarrow B = C/G$ be the quotient morphism.
- Then $K_C = \pi^*(K_B + \Delta)$, where $\Delta = \sum \frac{r_i-1}{r_i} p_i$ comes from Riemann-Hurwitz.
- $2g - 2 = \operatorname{vol}(K_C) = |G| \cdot \operatorname{vol}(K_B + \Delta) = |G|(2h - 2 + \sum \frac{r_i-1}{r_i})$.

Riemann-Hurwitz

- Let C be a smooth curve of genus g with automorphism group G and let $\pi\colon C \longrightarrow B = C/G$ be the quotient morphism.
- Then $K_C = \pi^*(K_B + \Delta)$, where $\Delta = \sum \frac{r_i - 1}{r_i} p_i$ comes from Riemann-Hurwitz.
- $2g - 2 = \operatorname{vol}(K_C) = |G| \cdot \operatorname{vol}(K_B + \Delta) = |G|(2h - 2 + \sum \frac{r_i - 1}{r_i})$.
- So we want to bound the quantity $2h - 2 + \sum \frac{r_i - 1}{r_i}$, the volume of $K_B + \Delta$, from below.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i - 1}{r_i} > 0$, so that $v \geq 1/2$.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i - 1}{r_i} > 0$, so that $v \geq 1/2$.
- Otherwise $h = 0$ and $\sum \frac{r_i - 1}{r_i} > 2$.

A lower bound for $v = 2h-2+\sum \frac{r_i-1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i-1}{r_i} > 0$, so that $v \geq 1/2$.
- Otherwise $h = 0$ and $\sum \frac{r_i-1}{r_i} > 2$.
- Five terms: $v \geq 5 \cdot 1/2 - 2 = 1/2$.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i - 1}{r_i} > 0$, so that $v \geq 1/2$.
- Otherwise $h = 0$ and $\sum \frac{r_i - 1}{r_i} > 2$.
- Five terms: $v \geq 5 \cdot 1/2 - 2 = 1/2$.
- Four terms: $v \geq 3 \cdot 1/2 + 2/3 - 2 = 1/6$.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i - 1}{r_i} > 0$, so that $v \geq 1/2$.
- Otherwise $h = 0$ and $\sum \frac{r_i - 1}{r_i} > 2$.
- Five terms: $v \geq 5 \cdot 1/2 - 2 = 1/2$.
- Four terms: $v \geq 3 \cdot 1/2 + 2/3 - 2 = 1/6$.
- Three terms: $v \geq 1/2 + 2/3 + 6/7 - 2 = 1/42$.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i - 1}{r_i} > 0$, so that $v \geq 1/2$.
- Otherwise $h = 0$ and $\sum \frac{r_i - 1}{r_i} > 2$.
- Five terms: $v \geq 5 \cdot 1/2 - 2 = 1/2$.
- Four terms: $v \geq 3 \cdot 1/2 + 2/3 - 2 = 1/6$.
- Three terms: $v \geq 1/2 + 2/3 + 6/7 - 2 = 1/42$.
- So we get equality if and only if there is a Riemann surface which is a cover of $B = \mathbb{P}^1$, ramified over 0, 1 and ∞ to orders 2, 3 and 7.

A lower bound for $v = 2h - 2 + \sum \frac{r_i - 1}{r_i}$.

- If $h \geq 2$ then $v \geq 2$.
- If $h = 1$ then $\sum \frac{r_i - 1}{r_i} > 0$, so that $v \geq 1/2$.
- Otherwise $h = 0$ and $\sum \frac{r_i - 1}{r_i} > 2$.
- Five terms: $v \geq 5 \cdot 1/2 - 2 = 1/2$.
- Four terms: $v \geq 3 \cdot 1/2 + 2/3 - 2 = 1/6$.
- Three terms: $v \geq 1/2 + 2/3 + 6/7 - 2 = 1/42$.
- So we get equality if and only if there is a Riemann surface which is a cover of $B = \mathbb{P}^1$, ramified over 0, 1 and ∞ to orders 2, 3 and 7.
- This is a purely topological question.

Lower bound

- We can run the same argument in all dimensions.

Lower bound

- We can run the same argument in all dimensions.
- We need to bound the volume of a log smooth pair (X, Δ) from below, where X is a projective variety and the coefficients of Δ are of the form $(r-1)/r$.

Lower bound

- We can run the same argument in all dimensions.
- We need to bound the volume of a log smooth pair (X, Δ) from below, where X is a projective variety and the coefficients of Δ are of the form $(r-1)/r$.

heorem: (Hacon,-,Xu) Suppose that $I \subset [0,1]$ satisfies the DCC and $\bar{I} \subset \mathbb{Q}$. Let $\mathfrak{D}$ denote the set of all log smooth pairs (X, Δ) such that the coefficients of Δ belong to I. Then the set

$$\{\,\mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D}\,\},$$

satisfies the DCC.

Lower bound

- We can run the same argument in all dimensions.
- We need to bound the volume of a log smooth pair (X, Δ) from below, where X is a projective variety and the coefficients of Δ are of the form $(r-1)/r$.

heorem: (Hacon,-,Xu) Suppose that $I \subset [0,1]$ satisfies the DCC and $\bar{I} \subset \mathbb{Q}$. Let $\mathfrak{D}$ denote the set of all log smooth pairs (X, Δ) such that the coefficients of Δ belong to I. Then the set

$$\{\,\mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D}\,\},$$

satisfies the DCC.

Status: A paper in the case of a global quotient will appear soon and a paper containing the general case exists.

Descending Chain Condition

- Note that the set

$$R = \left\{ \frac{r-1}{r} \,\middle|\, r \in \mathbb{N} \cup \{\infty\} \right\},$$

satisfies the DCC, and the only accumulation point is one.

Descending Chain Condition

- Note that the set

$$R = \{\, \frac{r-1}{r} \,|\, r \in \mathbb{N} \cup \{\infty\} \,\},$$

  satisfies the DCC, and the only accumulation point is one.
- Note that if the set

$$\{\, \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

  satisfies the DCC, then there is a constant $\delta > 0$ such that if $\operatorname{vol}(X, K_X + \Delta) \neq 0$, then $\operatorname{vol}(X, K_X + \Delta) > \delta$.

Descending Chain Condition

- Note that the set
$$R = \{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \cup \{\infty\} \},$$
satisfies the DCC, and the only accumulation point is one.
- Note that if the set
$$\{ \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \},$$
satisfies the DCC, then there is a constant $\delta > 0$ such that if $\operatorname{vol}(X, K_X + \Delta) \neq 0$, then $\operatorname{vol}(X, K_X + \Delta) > \delta$.
- In the case when $I = R$, $c = 1/\delta$ is an upper bound.

The argument of Tsuji

- The proof proceeds by induction.

The argument of Tsuji

- The proof proceeds by induction.
- We prove that there is a constant M, which only depends on I such that the natural rational map $\phi_{M(K_X+\Delta)}$ associated to $|M(K_X+\Delta)|$ is birational.

The argument of Tsuji

- The proof proceeds by induction.
- We prove that there is a constant M, which only depends on I such that the natural rational map $\phi_{M(K_X+\Delta)}$ associated to $|M(K_X+\Delta)|$ is birational.
- The hard part is to deal with the case when the volume is small.

The argument of Tsuji

- The proof proceeds by induction.
- We prove that there is a constant M, which only depends on I such that the natural rational map $\phi_{M(K_X+\Delta)}$ associated to $|M(K_X + \Delta)|$ is birational.
- The hard part is to deal with the case when the volume is small.
- In fact, one first proves that $\phi_{M(K_X+\Delta)}$ is birational if $\mathrm{vol}(X, M(K_X + \Delta))$ is bounded from below.

The argument of Tsuji

- The proof proceeds by induction.
- We prove that there is a constant M, which only depends on I such that the natural rational map $\phi_{M(K_X+\Delta)}$ associated to $|M(K_X+\Delta)|$ is birational.
- The hard part is to deal with the case when the volume is small.
- In fact, one first proves that $\phi_{M(K_X+\Delta)}$ is birational if $\operatorname{vol}(X, M(K_X+\Delta))$ is bounded from below.
- This step is hard and uses lifting of sections and comparison of various types of adjunction.

The argument of Tsuji

- The proof proceeds by induction.
- We prove that there is a constant M, which only depends on I such that the natural rational map $\phi_{M(K_X+\Delta)}$ associated to $|M(K_X+\Delta)|$ is birational.
- The hard part is to deal with the case when the volume is small.
- In fact, one first proves that $\phi_{M(K_X+\Delta)}$ is birational if $\operatorname{vol}(X, M(K_X+\Delta))$ is bounded from below.
- This step is hard and uses lifting of sections and comparison of various types of adjunction.
- This step is much easier in the case when (X, Δ) is a quotient.

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that given any $(X, \Delta) \in \mathfrak{D}$, there is a $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$, where the strict transform of Δ, and

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that given any $(X, \Delta) \in \mathfrak{D}$, there is a $t \in T$ and a birational map $f\colon Z_t \dashrightarrow X$, where the strict transform of Δ, and the exceptional locus of f_t is contained in the support of B_t.

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that given any $(X, \Delta) \in \mathfrak{D}$, there is a $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$, where the strict transform of Δ, and the exceptional locus of f_t is contained in the support of B_t.

- the family of rational surfaces, with empty boundary, is log birationally bounded.

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that given any $(X, \Delta) \in \mathfrak{D}$, there is a $t \in T$ and a birational map $f\colon Z_t \dashrightarrow X$, where the strict transform of Δ, and the exceptional locus of f_t is contained in the support of B_t.

- the family of rational surfaces, with empty boundary, is log birationally bounded. In fact take T a point, and $Z = (\mathbb{P}^2, L)$.

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that given any $(X, \Delta) \in \mathfrak{D}$, there is a $t \in T$ and a birational map $f : Z_t \dashrightarrow X$, where the strict transform of Δ, and the exceptional locus of f_t is contained in the support of B_t.

- the family of rational surfaces, with empty boundary, is log birationally bounded. In fact take T a point, and $Z = (\mathbb{P}^2, L)$.
- the set of rational threefolds is not log birationally bounded.

Log birationally bounded

finition: We will say that a family of log pairs $\mathfrak{D}$ is log birationally bounded, if there is a flat family of log pairs $(Z, B) \longrightarrow T$, T of finite type, such that given any $(X, \Delta) \in \mathfrak{D}$, there is a $t \in T$ and a birational map $f\colon Z_t \dashrightarrow X$, where the strict transform of Δ, and the exceptional locus of f_t is contained in the support of B_t.

- the family of rational surfaces, with empty boundary, is log birationally bounded. In fact take T a point, and $Z = (\mathbb{P}^2, L)$.
- the set of rational threefolds is not log birationally bounded. Just take $\mathbb{P}^2 \times \mathbb{P}^1$ and blow up $C \times \{0\}$, where C is a smooth plane curve.

The argument of Alexeev

heorem: (Hacon,-,Xu) If $\mathfrak{D}$ is a log birationally bounded family of log pairs, where the coefficients of Δ belong to a set I which satisfies the DCC, then the set

$$\{\, \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

satisfies the DCC.

The argument of Alexeev

heorem: (Hacon,-,Xu) If $\mathfrak{D}$ is a log birationally bounded family of log pairs, where the coefficients of Δ belong to a set I which satisfies the DCC, then the set

$$\{\, \mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

satisfies the DCC.

- If (X, Δ) is kawamata log terminal and the coefficients belong to a finite set, then the volume takes on only finitely many values.

The argument of Alexeev

heorem: (Hacon,-,Xu) If $\mathfrak{D}$ is a log birationally bounded family of log pairs, where the coefficients of Δ belong to a set I which satisfies the DCC, then the set

$$\{\, \mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

satisfies the DCC.

- If (X, Δ) is kawamata log terminal and the coefficients belong to a finite set, then the volume takes on only finitely many values.
- One possible application is to boundedness of the moduli functor of varieties of general type.

The argument of Alexeev

heorem: (Hacon,-,Xu) If $\mathfrak{D}$ is a log birationally bounded family of log pairs, where the coefficients of Δ belong to a set I which satisfies the DCC, then the set

$$\{\,\mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D}\,\},$$

satisfies the DCC.

- If (X, Δ) is kawamata log terminal and the coefficients belong to a finite set, then the volume takes on only finitely many values.
- One possible application is to boundedness of the moduli functor of varieties of general type.
- In fact Alexeev proved stronger statements for all of these results in the case of surfaces.

ACC for the log canonical threshold

- We sketch another possible application.

ACC for the log canonical threshold

- We sketch another possible application.
- First we give some background.

ACC for the log canonical threshold

- We sketch another possible application.
- First we give some background.
- Let $C \subset \mathbb{C}^2$ be a plane curve. The log canonical threshold λ of C is the largest real number t such that $(\mathbb{C}^2, tC)$ is log canonical.

ACC for the log canonical threshold

- We sketch another possible application.
- First we give some background.
- Let $C \subset \mathbb{C}^2$ be a plane curve. The log canonical threshold λ of C is the largest real number t such that $(\mathbb{C}^2, tC)$ is log canonical.
- This means that there is a divisor, of log discrepancy zero, that is, coefficient one.

ACC for the log canonical threshold

- We sketch another possible application.
- First we give some background.
- Let $C \subset \mathbb{C}^2$ be a plane curve. The log canonical threshold λ of C is the largest real number t such that $(\mathbb{C}^2, tC)$ is log canonical.
- This means that there is a divisor, of log discrepancy zero, that is, coefficient one.
- In particular $\lambda \in (0, 1]$, since C has coefficient one.

ACC for the log canonical threshold

- We sketch another possible application.
- First we give some background.
- Let $C \subset \mathbb{C}^2$ be a plane curve. The log canonical threshold λ of C is the largest real number t such that $(\mathbb{C}^2, tC)$ is log canonical.
- This means that there is a divisor, of log discrepancy zero, that is, coefficient one.
- In particular $\lambda \in (0, 1]$, since C has coefficient one.
- Note that the smaller the log canonical threshold, the worse the singularities.

Some examples

- If C is given by $y^2 + x^3$, then the log canonical threshold is $5/6$ and the exceptional divisor of log discrepancy zero is given by the last exceptional divisor of the minimal log resolution.

Some examples

- If C is given by $y^2 + x^3$, then the log canonical threshold is $5/6$ and the exceptional divisor of log discrepancy zero is given by the last exceptional divisor of the minimal log resolution.
- If $\pi\colon S \longrightarrow \mathbb{C}^2$ extracts just this exceptional divisor then we have $K_S + 5/6\tilde{C} + E = \pi^*(K_{\mathbb{C}^2} + 5/6C)$.

Some examples

- If C is given by $y^2 + x^3$, then the log canonical threshold is $5/6$ and the exceptional divisor of log discrepancy zero is given by the last exceptional divisor of the minimal log resolution.
- If $\pi\colon S \longrightarrow \mathbb{C}^2$ extracts just this exceptional divisor then we have $K_S + 5/6\tilde{C} + E = \pi^*(K_{\mathbb{C}^2} + 5/6C)$.
- If we restrict to E we get a numerically trivial divisor, and we may think of this as giving us an equation for a, $(K_S + a\tilde{C} + E) \cdot E = 0$.

Some examples

- If C is given by $y^2 + x^3$, then the log canonical threshold is $5/6$ and the exceptional divisor of log discrepancy zero is given by the last exceptional divisor of the minimal log resolution.
- If $\pi\colon S \longrightarrow \mathbb{C}^2$ extracts just this exceptional divisor then we have $K_S + 5/6\tilde{C} + E = \pi^*(K_{\mathbb{C}^2} + 5/6C)$.
- If we restrict to E we get a numerically trivial divisor, and we may think of this as giving us an equation for a, $(K_S + a\tilde{C} + E) \cdot E = 0$.
- Now S has two singularities along E, one of index two and the other of index three, so that $(K_S + a\tilde{C} + E)|_E = -2 + 1/2 + 2/3 + a$ and $a = 5/6$, by orbifold adjunction.

Further examples

- If C is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, by a very similar calculation.

Further examples

- If C is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, by a very similar calculation.
- More generally still, if $S \subset \mathbb{P}^n$ is the hypersurface given by $x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ then the log canonical threshold is $\min(1/a_1 + 1/a_2 + \cdots + 1/a_n, 1)$.

Further examples

- If C is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, by a very similar calculation.
- More generally still, if $S \subset \mathbb{P}^n$ is the hypersurface given by $x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ then the log canonical threshold is $\min(1/a_1 + 1/a_2 + \cdots + 1/a_n, 1)$.

jecture: (Shokurov) The set of all log canonical thresholds satisfies the ACC.

Further examples

- If C is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, by a very similar calculation.
- More generally still, if $S \subset \mathbb{P}^n$ is the hypersurface given by $x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ then the log canonical threshold is $\min(1/a_1 + 1/a_2 + \cdots + 1/a_n, 1)$.

jecture: (Shokurov) The set of all log canonical thresholds satisfies the ACC.

heorem: (de Fernex, Ein, Kollár, Mustaţă) This conjecture holds for hypersurfaces.

Further examples

- If C is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, by a very similar calculation.
- More generally still, if $S \subset \mathbb{P}^n$ is the hypersurface given by $x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ then the log canonical threshold is $\min(1/a_1 + 1/a_2 + \cdots + 1/a_n, 1)$.

jecture: (Shokurov) The set of all log canonical thresholds satisfies the ACC.

heorem: (de Fernex, Ein, Kollár, Mustaţă) This conjecture holds for hypersurfaces.

- We hope to prove the full version of Shokurov's conjecture using birational boundedness.

Inductive arguments

- If a set satisfies the ACC, we may run induction.

Inductive arguments

- If a set satisfies the ACC, we may run induction.

heorem: (Birkar) Assume termination of all flips in dimension $n-1$ and ACC for the log canonical threshold in dimension n.
If $K_X + \Delta$ is kawamata log terminal and $K_X + \Delta$ is numerically equivalent to $D \geq 0$, then any sequence of $(K_X + \Delta)$-flips terminates.

Inductive arguments

- If a set satisfies the ACC, we may run induction.

heorem: (Birkar) Assume termination of all flips in dimension $n-1$ and ACC for the log canonical threshold in dimension n.
If $K_X+\Delta$ is kawamata log terminal and $K_X+\Delta$ is numerically equivalent to $D \geq 0$, then any sequence of $(K_X+\Delta)$-flips terminates.

- It is natural to wonder what are the accumulation points of any set which satisfies the ACC.

Inductive arguments

- If a set satisfies the ACC, we may run induction.

heorem: (Birkar) Assume termination of all flips in dimension $n-1$ and ACC for the log canonical threshold in dimension n.
If $K_X+\Delta$ is kawamata log terminal and $K_X+\Delta$ is numerically equivalent to $D \geq 0$, then any sequence of $(K_X+\Delta)$-flips terminates.

- It is natural to wonder what are the accumulation points of any set which satisfies the ACC.

jecture: (Kollár) Any accumulation point of the log canonical threshold in dimension n is a log canonical threshold in dimension $n-1$. In particular, the set of accumulation points is rational.

Standard reduction

- First we extract a divisor of log discrepancy zero (coefficient one) and restrict to this divisor.

Standard reduction

- First we extract a divisor of log discrepancy zero (coefficient one) and restrict to this divisor.
- We are reduced to considering log canonical pairs (X, Δ), such that $K_X + \Delta$ is numerically trivial, in dimension one less.

Standard reduction

- First we extract a divisor of log discrepancy zero (coefficient one) and restrict to this divisor.
- We are reduced to considering log canonical pairs (X, Δ), such that $K_X + \Delta$ is numerically trivial, in dimension one less.
- If the conjecture fails, then there is an infinite sequence (X_i, Δ_i) whose coefficients are increasing.

Standard reduction

- First we extract a divisor of log discrepancy zero (coefficient one) and restrict to this divisor.
- We are reduced to considering log canonical pairs (X, Δ), such that $K_X + \Delta$ is numerically trivial, in dimension one less.
- If the conjecture fails, then there is an infinite sequence (X_i, Δ_i) whose coefficients are increasing.
- Running the MMP, we are reduced to the case when the Picard number of X is one.

Standard reduction

- First we extract a divisor of log discrepancy zero (coefficient one) and restrict to this divisor.
- We are reduced to considering log canonical pairs (X, Δ), such that $K_X + \Delta$ is numerically trivial, in dimension one less.
- If the conjecture fails, then there is an infinite sequence (X_i, Δ_i) whose coefficients are increasing.
- Running the MMP, we are reduced to the case when the Picard number of X is one.
- Shifting the coefficients around, we may assume that there is only one component whose coefficient is increasing.

Warm up

- Suppose that (X_i, Δ_i) is not kawamata log terminal.

Warm up

- Suppose that (X_i, Δ_i) is not kawamata log terminal.
- Extracting a divisor of log discrepancy zero (coefficient one), running the MMP, and restricting to this component, we reduce the dimension by one, and we are done by induction.

Warm up

- Suppose that (X_i, Δ_i) is not kawamata log terminal.
- Extracting a divisor of log discrepancy zero (coefficient one), running the MMP, and restricting to this component, we reduce the dimension by one, and we are done by induction.
- In fact our aim is to either reduce to this case, or argue that we are already done.

Warm up

- Suppose that (X_i, Δ_i) is not kawamata log terminal.
- Extracting a divisor of log discrepancy zero (coefficient one), running the MMP, and restricting to this component, we reduce the dimension by one, and we are done by induction.
- In fact our aim is to either reduce to this case, or argue that we are already done.
- The standard method to create non kawamata log terminal centres is the method of concentration of singularities due to Kawamata and Shokurov.

Warm up

- Suppose that (X_i, Δ_i) is not kawamata log terminal.
- Extracting a divisor of log discrepancy zero (coefficient one), running the MMP, and restricting to this component, we reduce the dimension by one, and we are done by induction.
- In fact our aim is to either reduce to this case, or argue that we are already done.
- The standard method to create non kawamata log terminal centres is the method of concentration of singularities due to Kawamata and Shokurov.
- For this to work we need a divisor of large volume.

Two cases

Δ_i^n is unbounded:

Two cases

Δ_i^n is unbounded: There is a divisor $D_i \sim_{\mathbb{R}} \epsilon_i \Delta_i$ such that $K_{X_i} + (1-\epsilon_i)\Delta_i + D_i$ is not kawamata log terminal,

Two cases

Δ_i^n is unbounded: There is a divisor $D_i \sim_{\mathbb{R}} \epsilon_i \Delta_i$ such that $K_{X_i} + (1-\epsilon_i)\Delta_i + D_i$ is not kawamata log terminal, and we are done by a similar argument to the warm up argument.

Two cases

Δ_i^n is unbounded: There is a divisor $D_i \sim_{\mathbb{R}} \epsilon_i \Delta_i$ such that $K_{X_i} + (1-\epsilon_i)\Delta_i + D_i$ is not kawamata log terminal, and we are done by a similar argument to the warm up argument.

Otherwise Δ_i^n is bounded:

Two cases

Δ_i^n is unbounded: There is a divisor $D_i \sim_{\mathbb{R}} \epsilon_i \Delta_i$ such that $K_{X_i} + (1 - \epsilon_i)\Delta_i + D_i$ is not kawamata log terminal, and we are done by a similar argument to the warm up argument.

Otherwise Δ_i^n is bounded: Let Λ_i be the same divisor as Δ_i, but with the limiting coefficients.

Two cases

Δ_i^n is unbounded: There is a divisor $D_i \sim_{\mathbb{R}} \epsilon_i \Delta_i$ such that $K_{X_i} + (1 - \epsilon_i)\Delta_i + D_i$ is not kawamata log terminal, and we are done by a similar argument to the warm up argument.

Otherwise Δ_i^n is bounded: Let Λ_i be the same divisor as Δ_i, but with the limiting coefficients. By induction, we may find $\Lambda_i \leq \Pi_i$, with fixed rational coefficients, such that $K_{X_i} + \Pi_i$ is klt.

Two cases

Δ_i^n is unbounded: There is a divisor $D_i \sim_{\mathbb{R}} \epsilon_i \Delta_i$ such that $K_{X_i} + (1 - \epsilon_i)\Delta_i + D_i$ is not kawamata log terminal, and we are done by a similar argument to the warm up argument.

Otherwise Δ_i^n is bounded: Let Λ_i be the same divisor as Δ_i, but with the limiting coefficients. By induction, we may find $\Lambda_i \leq \Pi_i$, with fixed rational coefficients, such that $K_{X_i} + \Pi_i$ is klt.

$K_{X_i} + \Pi_i$ has bounded volume, so this family is log birationally bounded and the result is clear in this case.

Idle speculation

jecture: (Borisov,Alexeev,Borisov) Fix n and $\epsilon > 0$.
The family of all Fano varieties of dimension n and log discrepancy at least $\epsilon > 0$ is bounded (in the usual sense).

Idle speculation

jecture: (Borisov,Alexeev,Borisov) Fix n and $\epsilon > 0$.
The family of all Fano varieties of dimension n and log discrepancy at least $\epsilon > 0$ is bounded (in the usual sense).

- It has long been realised that this conjecture implies many other conjectures, such as ACC for the log canonical threshold and Batyrev's conjecture, to do with the cone of nef curves.

Idle speculation

jecture: (Borisov,Alexeev,Borisov) Fix n and $\epsilon > 0$. The family of all Fano varieties of dimension n and log discrepancy at least $\epsilon > 0$ is bounded (in the usual sense).

- It has long been realised that this conjecture implies many other conjectures, such as ACC for the log canonical threshold and Batyrev's conjecture, to do with the cone of nef curves.

uestion: Perhaps one can push birational boundedness methods to prove some of these conjectures, even the BAB conjecture?

Idle speculation

jecture: (Borisov,Alexeev,Borisov) Fix n and $\epsilon > 0$. The family of all Fano varieties of dimension n and log discrepancy at least $\epsilon > 0$ is bounded (in the usual sense).

- It has long been realised that this conjecture implies many other conjectures, such as ACC for the log canonical threshold and Batyrev's conjecture, to do with the cone of nef curves.

uestion: Perhaps one can push birational boundedness methods to prove some of these conjectures, even the BAB conjecture?

uestion: Perhaps one can prove termination of flips for $K_X + \Delta$ kawamata log terminal and Δ big?