ON THE EXISTENCE OF FLIPS

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ABSTRACT. Using the techniques of [17], [7], [13] and [15], we prove that flips exist in dimension n, if one assumes the termination of real flips in dimension $n-1$.

1. Introduction

The main result of this paper is:

Theorem 1.1. *Assume the real MMP in dimension $n-1$.*
Then flips exist in dimension n.

Here are two consequences of this result:

Corollary 1.2. *Assume termination of real flips in dimension $n-1$ and termination of flips in dimension n.*
Then the MMP exists in dimension n.

As Shokurov has proved, [14], the termination of real flips in dimension three, we get a new proof of the following result of Shokurov [15]:

Corollary 1.3. *Flips exist in dimension four.*

Given a proper variety, it is natural to search for a good birational model. An obvious, albeit hard, first step is to pick a smooth projective model. Unfortunately there are far too many such models; indeed given any such, we can construct infinitely many more, simply by virtue of successively blowing up smooth subvarieties. To construct a unique model, or at least cut down the examples to a manageable number, we have to impose some sort of minimality on the birational model.

The choice of such a model depends on the global geometry of X. One possibility is that we can find a model on which the canonical divisor K_X is nef, so that its intersection with any curve is non-negative.

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Conjecturally this is equivalent to the condition that the Kodaira dimension of X is non-negative, that is there are global pluricanonical forms. Another possibility is that through any point of X there passes a rational curve. In this case the canonical divisor is certainly negative on such a curve, and the best one can hope for is that there is a fibration on which the anticanonical divisor is relatively ample. In other words we are searching for a birational model X such that either

(1) K_X is nef, in which case X is called a *minimal model*, or
(2) there is a fibration $X \longrightarrow Z$ of relative Picard number one, such that $-K_X$ is relatively ample; we call this a *Mori fibre space*.

The minimal model program is an attempt to construct such a model step by step. We start with a smooth projective model X. If K_X is nef, then we have case (1). Otherwise the cone theorem guarantees the existence of a contraction morphism $f : X \longrightarrow Z$ of relative Picard number one, such that $-K_X$ is relatively ample. If the dimension of Z is less than the dimension of X, then we have case (2). Otherwise f is birational. If f is divisorial (that is the exceptional locus is a divisor), then we are free to replace X by Z and continue this process. Even though Z may be singular, it is not hard to prove that it is $\mathbb{Q}$-factorial, so that any Weil divisor is $\mathbb{Q}$-Cartier (some multiple is Cartier), and that X has terminal singularities. In particular it still makes sense to ask whether K_X is nef, and the cone theorem still applies at this level of generality. The tricky case is when f is not divisorial, since in this case it is not hard to show that no multiple of K_Z is Cartier, and it no longer even makes sense to ask if K_Z is nef. At this stage we have to construct the flip.

Let $f : X \longrightarrow Z$ be a small projective morphism of normal varieties, so that f is birational but does not contract any divisors. If D is any $\mathbb{Q}$-Cartier divisor such that $-D$ is relatively ample the *flip* of D, if it exists at all, is a commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Z, & \end{array}$$

where $X \dashrightarrow X'$ is birational, and the strict transform D' of D is relatively ample. Note that f' is unique, if it exists at all; indeed if we set

$$\mathfrak{R} = R(X, D) = \bigoplus_{n \in \mathbb{N}} f_* \mathcal{O}_X(nD),$$

then

$$X' = \operatorname{Proj}_Z \mathfrak{R}.$$

In particular the existence of the flip is equivalent to finite generation of the ring $\mathfrak{R}$.

It is too much to expect the existence of general flips; we do however expect that flips exist if $D = K_X + \Delta$ is kawamata log terminal. Supposing that the flip of $D = K_X$ exists, we replace X by X' and continue. Unfortunately this raises another issue, how do we know that this process terminates? It is clear that we cannot construct an infinite sequence of divisorial contractions, since the Picard number drops by one after every divisorial contraction, and the Picard number of X is finite. In other words, to establish the existence of the MMP, it suffices to prove the existence and termination of flips. Thus (1.1) reduces the existence of the MMP in dimension n, to termination of flips in dimension n.

In the following two conjectures, we work with either the field $K = \mathbb{Q}$ or $\mathbb{R}$.

Conjecture 1.4 (Existence of Flips). *Let (X, Δ) be a kawamata log terminal $\mathbb{Q}$-factorial pair of dimension n, where Δ is a K-divisor. Let $f\colon X \longrightarrow Z$ be a flipping contraction, so that $-(K_X + \Delta)$ is relatively ample, and f is a small contraction of relative Picard number one.*

Then the flip of f exists.

Conjecture 1.5 (Termination of Flips). *Let (X, Δ) be a kawamata log terminal $\mathbb{Q}$-factorial pair of dimension n, where Δ is a K-divisor.*

Then there is no infinite sequence of $(K_X + \Delta)$-flips.

For us, the statement "assuming the (real) MMP in dimension n" means precisely assuming $(1.4)_{\mathbb{Q},n}$ and $(1.5)_{\mathbb{R},n}$. In fact it is straightforward to see that $(1.4)_{\mathbb{Q},n}$ implies $(1.4)_{\mathbb{R},n}$, see for example the proof of (7.2).

Mori, in a landmark paper [12], proved the existence of 3-fold flips, when X is terminal and Δ is empty. Later on Shokurov [13] and Kollár [9] proved the existence of 3-fold flips for kawamata log terminal pairs (X, Δ), that is they proved $(1.4)_3$. Much more recently [15], Shokurov proved $(1.4)_4$.

Kawamata, [6], proved the termination of any sequence of threefold, kawamata log terminal flips, that is he proved $(1.5)_{\mathbb{Q},3}$. As previously pointed out, Shokurov proved, [14], $(1.5)_{\mathbb{R},3}$. Further, Shokurov proved, [16], that $(1.5)_{\mathbb{R},n}$ follows from two conjectures on the behaviour of the log discrepancy of pairs (X, Δ) of dimension n (namely acc for the set of log discrepancies, whenever the coefficients of Δ are confined to belong to a set of real numbers which satisfies dcc, and semicontinuity of the log discrepancy). Finally, Birkar, in a very recent preprint, [1], has reduced $(1.5)_{\mathbb{R},n}$, in the case when $K_X + \Delta$ has non-negative Kodaira dimension, to acc for the log canonical threshold and the existence of the MMP in dimension $n-1$.

We also recall the abundance conjecture,

Conjecture 1.6 (Abundance). *Let (X, Δ) be a kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let $\pi \colon X \longrightarrow Z$ be a proper morphism, where Z is affine and normal.*

If $K_X + \Delta$ is nef, then it is semiample.

Note that the three conjectures, existence and termination of flips, and abundance, are the three most important conjectures in the MMP. For example, Kawamata proved, [5], that these three results imply additivity of Kodaira dimension.

Our proof of (1.1) follows the general strategy of [15]. The first key step was already established in [13], see also [9] and [3]. In fact it suffices to prove the existence of the flip for $D = K_X + S + B$, see (4.2), where S has coefficient one, and $K_X + S + B$ is purely log terminal. The key point is that this allows us to restrict to S, and we can try to apply induction. By adjunction we may write

$$(K_X + S + B)|_S = K_S + B',$$

where B' is effective and $K_S + B'$ is kawamata log terminal. In fact, since we are trying to prove finite generation of the ring $\mathfrak{R}$, the key point is to consider the restriction maps

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \longrightarrow H^0(X, \mathcal{O}_S(m(K_S + B'))),$$

see (3.5). Here and often elsewhere, we will assume that Z is affine, so that we can replace f_* by H^0. Now if these maps were surjective, we would be done by induction. Unfortunately this is too much to expect. However we are able to prove, after changing models, that something close to this does happen.

The starting point is to use the extension result proved in [4], which in turn builds on the work of Siu [17] and Kawamata [7]. To apply this result, we need to improve how S sits inside X. To this end, we pass to a resolution $Y \longrightarrow X$. Let T be the strict transform of S and let Γ be those divisors of log discrepancy less than one. Then by a generalisation of (3.17) of [4], we can extend sections from T to Y, provided that we can find $G \in |m(K_Y + \Gamma)|$ which does not contain any log canonical centre of $K_Y + \ulcorner \Gamma \urcorner$.

In fact if we blow up more, we can separate all of the components of Γ, except the intersections with T. In this case, the condition on G becomes that it does not share any components with Γ. Thus, for each m we are able to cancel common components, and lift sections. Putting all of this together, see §5 and §6 for more details, we get a sequence of divisors $\Theta_\bullet$ on T, such that

$$i\Theta_i + j\Theta_j \leq (i+j)\Theta_{i+j},$$

and it suffices to prove that this sequence stabilises, that is

$$\Theta_m = \Theta,$$

is constant for m sufficiently large and divisible. To this end, we take the limit Θ of this sequence. Then $K_T + \Theta$ is kawamata log terminal, but in general since Θ is a limit, it has real coefficients, rather than just rational.

Now there are two ways in which the sequence $\Theta_\bullet$ might vary. By assumption each $K_T + \Theta_m$ is big and so there is some model $T_m \longrightarrow T$ on which the mobile part of $mk(K_T + \Theta_m)$ becomes semiample. The problem is that the model T_m depends on m. This is obviously an issue of birational geometry, and can only occur in dimensions two and higher. To get around this, we need to run the real MMP, see §7. Thus replacing T by a higher model, we may assume that the mobile part P_m of some fixed multiple $ml(K_T + \Theta_m)$ is in fact free, and that the limit of P_m/m is semiample.

The second way in which the sequence $\Theta_\bullet$ might vary is that we might get freeness of the mobile part of $mk(K_T + \Theta_m)$ on the same model, but Θ_m is still not constant. There are plenty of such examples, even on the curve $\mathbb{P}^1$. Fortunately Shokurov has already proved that this cannot happen, since the sequence $\Theta_\bullet$ satisfies a subtle asymptoptic saturation property, see §8 and §9.

Hopefully it is clear, from what we just said, the great debt our proof of (1.1) owes to the work of Kawamata, Siu and especially Shokurov. The material in §5 was inspired by the work of Siu [17] and Kawamata [7] on deformation invariance of plurigenera, and lifting sections using multiplier ideas sheaves. On the other hand, a key step is to use the reduction to pl flips, due to Shokurov contained in [13]. Moreover, we use many of the results and ideas contained in [15], especially the notion of a saturated algebra.

Since the proof of (1.1) is not very long, we have erred on the side of making the proofs as complete as possible. We also owe a great debt to the work of Ambro, Fujino, Takagi and especially Corti, who did such a good job of making the work of Shokurov more accessible.

In particular much of the material contained in §3 and §7-9 is due to Shokurov, as well as some of the material in the other sections, and we have followed the exposition of [2] and [3] quite closely.

2. Notation and conventions

We work over the field of complex numbers $\mathbb{C}$. A $\mathbb{Q}$-Cartier divisor D on a normal variety X is *nef* if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{Q}$-divisors D_1, D_2 are $\mathbb{Q}$-linearly equivalent ($D_1 \sim_{\mathbb{Q}} D_2$) if there exists an integer $m > 0$ such that mD_i are linearly equivalent. We say that a $\mathbb{Q}$-Weil divisor D is big if we may find an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor B, such that $D \sim_{\mathbb{Q}} A + B$. If T is a subvariety of X, not contained in the base locus of $|D|$, then $|D|_T$ denotes the image of the linear system $|D|$ under restriction to T.

A *log pair* (X, Δ) is a normal variety X and an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that a log pair (X, Δ) is *log smooth*, if X is smooth and the support of Δ is a divisor with global normal crossings. A projective morphism $g\colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if Y is smooth and $g^{-1}(\Delta) \cup \{\text{ exceptional set of } g\,\}$ is a divisor with normal crossings support. We write $g^*(K_X + \Delta) = K_Y + \Gamma$ and $\Gamma = \sum a_i \Gamma_i$ where Γ_i are distinct reduced irreducible divisors. The log discrepancy of Γ_i is $1 - a_i$. The *locus of log canonical singularities of the pair* (X, Δ), denoted $\mathrm{LCS}(X, \Delta)$, is equal to the image of those components of Γ of coefficient at least one (equivalently log discrepancy at most zero). The pair (X, Δ) is *kawamata log terminal* if for every (equivalently for one) log resolution $g\colon Y \longrightarrow X$ as above, the coefficients of Γ are strictly less than one, that is $a_i < 1$ for all i. Equivalently, the pair (X, Δ) is kawamata log terminal if the locus of log canonical singularities is empty. We say that the pair (X, Δ) is *purely log terminal* if the log discrepancy of any exceptional divisor is greater than zero.

We will also write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components, and E is g-exceptional. Note that this decomposition is unique.

Note that the group of Weil divisors with rational or real coefficients forms a vector space, with a canonical basis given by the prime divisors. Given an $\mathbb{R}$-divisor, $\|D\|$ denotes the sup norm with respect to this basis. We say that D' is *sufficiently close to* D if there is a finite dimensional vector space V such that D and $D' \in V$ and D' belongs to a sufficiently small ball of radius $\delta > 0$ about D,

$$\|D - D'\| < \delta.$$

If A and B are two $\mathbb{R}$-divisors, then we let $(A, B]$ denote the line segment

$$\{\, \lambda A + \mu B \mid \lambda + \mu = 1, \lambda \geq 0, \mu > 0 \,\},$$

and $\max(A, B)$ denotes the divisor obtained by taking the maximum coefficient by coefficient.

We recall some definitions involving divisors with real coefficients:

Definition 2.1. *Let X be a variety.*

(1) *An $\mathbb{R}$-**Weil divisor** D is an $\mathbb{R}$-linear combination of prime divisors.*
(2) *Two $\mathbb{R}$-divisors D and D' are $\mathbb{R}$-**linearly equivalent** if their difference is an $\mathbb{R}$-linear combination of principal divisors.*
(3) *An $\mathbb{R}$-**Cartier divisor** D is an $\mathbb{R}$-linear combination of Cartier divisors.*
(4) *An $\mathbb{R}$-Cartier divisor D is **ample** if it is $\mathbb{R}$-linearly equivalent to a positive linear combination of ample divisors (in the usual sense).*
(5) *An $\mathbb{R}$-divisor D is **effective** if it is a positive real linear combination of prime divisors.*
(6) *An $\mathbb{R}$-Cartier divisor D is **big** if it is $\mathbb{R}$-linearly equivalent to the sum of an ample divisor and an effective divisor.*
(7) *An $\mathbb{R}$-Cartier divisor D is **semiample** if there is a contraction $\pi\colon X \longrightarrow Y$ such that D is $\mathbb{R}$-linearly equivalent to the pullback of an ample divisor.*

Note that we may pullback $\mathbb{R}$-Cartier divisors, so that we may define the various flavours of log terminal and log canonical in the obvious way.

3. Generalities on Finite generation

In this section we give some of the basic definitions and results concerning finite generation; we only include the proofs for completeness.

We fix some notation. Let $f\colon X \longrightarrow Z$ be a projective morphism of normal varieties, where Z is affine. Let A be the coordinate ring of Z.

Definition-Lemma 3.1. *Let $\mathfrak{R}$ be any graded A-algebra. A **truncation** of $\mathfrak{R}$ is any A-algebra of the form*

$$\mathfrak{R}_{(d)} = \bigoplus_{m \in \mathbb{N}} \mathfrak{R}_{md},$$

for a positive integer d.

Then $\mathfrak{R}$ is finitely generated iff there is a positive integer d such that $\mathfrak{R}_{(d)}$ is finitely generated.

Proof. Suppose that $\mathfrak{R}$ is finitely generated. The cyclic group $\mathbb{Z}_d$ acts in an obvious way on $\mathfrak{R}$, and under this action $\mathfrak{R}_{(d)}$ is the algebra of invariants. Thus $\mathfrak{R}_{(d)}$ is finitely generated by Noether's Theorem, which says that the ring of invariants of a finitely generated ring, under the action of a finite group, is finitely generated.

Now suppose that $\mathfrak{R}_{(d)}$ is finitely generated. Let $f \in \mathfrak{R}$. Then f is a root of the monic polynomial

$$x^d - f^d \in \mathfrak{R}_{(d)}[x].$$

In particular $\mathfrak{R}$ is integral over $\mathfrak{R}_{(d)}$ and the result follows by Noether's Theorem on the finiteness of the integral closure. □

We are interested in finite generation of the following algebras:

Definition 3.2. *Let B be an integral Weil divisor on X. We call any $\mathcal{O}_Z$-algebra of the form*

$$\bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_X(mB),$$

***divisorial**.*

In particular we have:

Lemma 3.3. *Let X be a normal variety and let $\mathfrak{R}$ and $\mathfrak{R}'$ be two divisorial algebras associated to divisors D and D'.*

If $aD \sim a'D'$, where a and a' are positive integers, then $\mathfrak{R}$ is finitely generated iff $\mathfrak{R}'$ is finitely generated.

Proof. Clear, since $\mathfrak{R}$ and $\mathfrak{R}'$ have the same truncation. □

We want to restrict a divisorial algebra to a prime divisor S:

Definition 3.4. *Let $\mathfrak{R}$ be the divisorial algebra associated to the divisor B. The **restricted algebra** $\mathfrak{R}_S$ is the image*

$$\bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_X(mB) \longrightarrow \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_S(mB),$$

under the obvious restriction map.

Lemma 3.5. *If the algebra $\mathfrak{R}$ is finitely generated then so is the restricted algebra. Conversely, if S is linearly equivalent to a multiple of B, where B is an effective divisor which does not contain S and the restricted algebra is finitely generated, then so is $\mathfrak{R}$.*

Proof. Since by definition there is a surjective homomorphism

$$\phi\colon \mathfrak{R} \longrightarrow \mathfrak{R}_S,$$

it follows that if $\mathfrak{R}$ is finitely generated then so is $\mathfrak{R}_S$.

Now suppose that $S \sim bB$. Passing to a truncation, we may assume that $b = 1$. We can identify the space of sections of

$$\mathfrak{R}_m = f_*\mathcal{O}_X(mB)$$

with rational functions g, such that

$$(g) + mB \geq 0.$$

Let $g_1 \in \mathfrak{R}_1$ be a rational function such that

$$(g_1) + B = S.$$

Suppose we have $g \in \mathfrak{R}_m$, with $\phi(g) = 0$. Then the support of

$$(g) + mB,$$

contains S, so that we may write

$$(g) + mB = S + S',$$

where S' is effective. But then

$$(g) + mB = (g_1) + B + S',$$

so that

$$(g/g_1) + (m-1)B = S'.$$

But this says exactly that $g/g_1 \in \mathfrak{R}_{m-1}$, so that the kernel of ϕ is precisely the principal ideal generated by g_1. But then if $\mathfrak{R}_S$ is finitely generated, it is clear that $\mathfrak{R}$ is finitely generated. □

Definition 3.6. *We say that a sequence of $\mathbb{R}$-divisors $B_\bullet$ is* ***additive*** *if*

$$B_i + B_j \leq B_{i+j},$$

we say that it is ***convex*** *if*

$$\frac{i}{i+j}B_i + \frac{j}{i+j}B_j \leq B_{i+j},$$

and we say that it is ***bounded*** *if there is a divisor B such that*

$$B_i \leq B.$$

Since the maps in (3.4) are not in general surjective, the restricted algebra is not necessarily divisorial. However we will be able to show that it is of the following form:

Definition 3.7. *Any $\mathcal{O}_Z$-algebra of the form*

$$\bigoplus_{m\in\mathbb{N}} f_*\mathcal{O}_X(B_m),$$

where $B_\bullet$ is an additive sequence of integral Weil divisors, will be called ***geometric****.*

We are interested in giving necessary and sufficient conditions for a divisorial or more generally a geometric algebra to be finitely generated.

Definition 3.8. *Let B be an integral divisor on X, such that $h^0(X, \mathcal{O}_X(B)) \neq 0$. Let F be the* ***fixed part*** *of the linear system $|B|$, and set $M = B - F$. We may write*

$$|B| = |M| + F.$$

We call $M = \operatorname{Mob} B$ the ***mobile part*** *of B and we call $B = M + F$ the* ***decomposition*** *of B into its mobile and fixed part. We say that a divisor is* ***mobile*** *if the fixed part is empty.*

Note that M is not necessarily effective, but that $h^0(X, \mathcal{O}_X(M)) \neq 0$.

Definition 3.9. *Let $\mathfrak{R}$ be the geometric algebra associated to the additive sequence $B_\bullet$. Let*

$$B_m = M_m + F_m,$$

be the decomposition of B_m into its mobile and fixed parts. The sequence of divisors $M_\bullet$ is called the ***mobile sequence*** *and the sequence of $\mathbb{Q}$-divisors $D_\bullet$ given by*

$$D_i = \frac{M_i}{i},$$

is called the ***characteristic sequence****.*

We say that $\mathfrak{R}$ is ***semiample*** *if D is semiample, where D is the limit of the characteristic sequence.*

Clearly the mobile sequence is additive and the characteristic sequence is convex. The key point is that finite generation of a divisorial algebra only depends on the mobile part in each degree, even up to a birational map:

Lemma 3.10. *Let $\mathfrak{R}$ be a geometric algebra asociated to the additive sequence $B_\bullet$, where each B_i is $\mathbb{Q}$-Cartier. Let $g\colon Y \longrightarrow X$ be any birational morphism and let $\mathfrak{R}'$ be the geometric algebra on Y associated to an additive sequence $B'_\bullet$.*

Then $\mathfrak{R}$ is isomorphic to $\mathfrak{R}'$.

Proof. Clear. □

Lemma 3.11. *Let $\mathfrak{R}$ be a semiample geometric algebra and let D be the limit of the characteristic sequence.*

If $D = D_k$ for some positive integer k then $\mathfrak{R}$ is finitely generated.

Proof. Passing to a truncation, we may assume that $D = D_1$. But then

$$mD = mD_1 = mM_1 \leq M_m = mD_m \leq mD,$$

and so $D = D_m$, for all positive integers m. Possibly passing to another truncation, we may then assume that M_m is free for all m. Let $h\colon X \longrightarrow W$ be the contraction over Z associated to M_1, so that $M_1 = h^*H$, for some very ample divisor on W. We have $M_m = mM_1 = h^*(mH)$ and so the algebra $\mathfrak{R}$ is nothing more than the coordinate ring of W under the embedding of W in $\mathbb{P}^n$ given by H, which is easily seen to be finitely generated by Serre vanishing. □

4. Reduction to pl flips and finite generation

We recall the definition of a pl flipping contraction:

Definition 4.1. *We call a morphism $f\colon X \longrightarrow Z$ of normal varieties, where Z is affine, a* ***pl flipping contraction*** *if*

(1) *f is a small birational contraction of relative Picard number one,*
(2) *X is $\mathbb{Q}$-factorial,*
(3) *$K_X + \Delta$ is purely log terminal, where $S = \llcorner \Delta \lrcorner$ is irreducible, and*
(4) *$-(K_X + \Delta)$ and $-S$ are ample.*

Shokurov, [13], see also [9] and [3], has shown:

Theorem 4.2 (Shokurov)**.** *To prove* (1.1) *it suffices to construct the flip of a pl flipping contraction.*

The aim of the rest of the paper is to prove:

Theorem 4.3. *Let (X, Δ) be a log pair of dimension n and let $f\colon X \longrightarrow Z$ be a morphism, where Z is affine and normal. Let k be a positive integer such that $D = k(K_X + \Delta)$ is Cartier, and let $\mathfrak{R}$ be the divisorial algebra associated to D. Assume that*

(1) *$K_X + \Delta$ is purely log terminal,*
(2) *$S = \llcorner \Delta \lrcorner$ is irreducible,*
(3) *there is a divisor $G \in |D|$, such that S is not contained in the support of G,*
(4) *$\Delta - S \sim_{\mathbb{Q}} A + B$, where A is ample and B is an effective divisor, whose support does not contain S, and*
(5) *$-(K_X + \Delta)$ is ample.*

If the real MMP holds in dimension $n-1$ then the restricted algebra $\mathfrak{R}_S$ is finitely generated.

Note that the only interesting case of (4.3) is when f is birational, since otherwise the condition that $-(K_X + \Delta)$ is ample implies that $\kappa(X, K_X + \Delta) = -\infty$.

We note that to prove (1.1), it is sufficient to prove (4.3):

Lemma 4.4. $(4.3)_n$ *implies* $(1.1)_n$.

Proof. By (4.2) it suffices to prove the existence of pl flips. Since Z is affine and f is small, it follows that S is mobile. By (3.5) it follows that it suffices to prove that the restricted algebra is finitely generated. Hence it suffices to prove that a pl flip satisfies the hypothesis of (4.3). Properties (1-2) and (5) are automatic and (3) follows as S is mobile. $\Delta - S$ is automatically big, as f is birational, and so $\Delta - S \sim_{\mathbb{Q}} A + B$, where A is ample, and B is effective. As S is mobile, possibly replacing B by a $\mathbb{Q}$-linearly equivalent divisor, we may assume that B does not contain S. □

5. Extending sections

The key idea of the proof of (1.1) is to use the main result of [4] to lift sections. In this section, we show that we can improve this result, if we add some hypotheses. We recall some of the basic results about multiplier ideal sheaves.

Definition 5.1. *Let (X, Δ) be a log pair, where X is smooth and let $g\colon Y \longrightarrow X$ be a log resolution. Suppose that we write*

$$K_Y + \Gamma = g^*(K_X + \Delta).$$

The ***multiplier ideal sheaf*** *of the log pair (X, Δ) is defined as*

$$\mathcal{J}(X, \Delta) = \mathcal{J}(\Delta) = g_*\mathcal{O}_Y(-\llcorner \Gamma \lrcorner).$$

Note that the pair (X, Δ) is kawamata log terminal iff the multiplier ideal sheaf is equal to $\mathcal{O}_X$. Another key property of a multiplier ideal sheaf is that it is independent of the log resolution. Multiplier ideal sheaves have the following basic property, see (2.2.1) of [18]:

Lemma 5.2. *Let (X, Δ) be a kawamata log terminal pair, where X is a smooth variety, and let D be any $\mathbb{Q}$-divisor. Let $f\colon X \longrightarrow Z$ be any projective morphism, where Z is affine and normal. Let $\sigma \in H^0(X, L)$ be any section of a line bundle L, with zero locus $S \subset X$.*

If $D - S \leq \Delta$ then $\sigma \in H^0(X, L \otimes \mathcal{J}(D))$.

Proof. Let $g\colon Y \longrightarrow X$ be a log resolution of the pair $(X, D+\Delta)$. As S is integral

$$\llcorner g^*D \lrcorner - g^*S \leq \llcorner g^*\Delta \lrcorner,$$

and as the pair (X,Δ) is kawamata log terminal,

$$K_{Y/X} - \llcorner g^*\Delta \lrcorner = -\llcorner \Gamma \lrcorner \geq 0.$$

Thus

$$\begin{aligned} g^*\sigma &\in H^0(Y, g^*L(-g^*S)) \\ &\subset H^0(Y, g^*L(-g^*S + K_{Y/X} - \llcorner g^*\Delta \lrcorner)) \\ &\subset H^0(Y, g^*L(K_{Y/X} - \llcorner g^*D \lrcorner)). \end{aligned}$$

Pushing forward via g, we get

$$\sigma \in H^0(X, L \otimes \mathcal{J}(D)).$$

□

We also have the following important vanishing result, which is an easy consequence of Kawamata-Viehweg vanishing:

Theorem 5.3. *(Nadel Vanishing) Let X be a smooth variety, let Δ be an effective $\mathbb{Q}$-divisor. Let $f\colon X \longrightarrow Z$ be any projective morphism and let N be any integral divisor such that $N - \Delta$ is relatively big and nef.*

Then

$$R^i f_*(\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(\Delta)) = 0, \qquad \textit{for } i > 0.$$

Here is the main result of this section:

Theorem 5.4. *Let (Y,Γ) be a smooth log pair and let $\pi\colon Y \longrightarrow Z$ be a projective morphism, where Z is normal and affine. Let m be a positive integer, and let L be any line bundle on X, such that $c_1(L) \sim_{\mathbb{Q}} m(K_Y + \Gamma)$. Assume that*

1. *(Y,Γ) is purely log terminal,*
2. *$T = \llcorner \Gamma \lrcorner$ is irreducible,*
3. *$\Gamma - T \sim_{\mathbb{Q}} A + B$, where A is ample and B is an effective divisor, which does not contain T.*

Let $\Delta = (\Gamma - T)|_T$, so that

$$(K_Y + \Gamma)|_T = K_T + \Delta.$$

Suppose that there is an effective divisor H, which does not contain T, such that for every sufficiently divisible positive integer s, the image of the natural homomorphism

$$H^0(Y, L^{\otimes s}(H)) \longrightarrow H^0(T, L^{\otimes s}(H)|_T),$$

contains the image of $H^0(T, L^{\otimes s}|_T)$, considered as a subspace of $H^0(T, L^{\otimes s}(H)|_T)$ by the inclusion induced by H.

Then the natural restriction homomorphism

$$H^0(Y, L) \longrightarrow H^0(T, L|_T),$$

is surjective.

Proof. As $K_Y + T + (1-\epsilon)(\Gamma - T) + \epsilon A + \epsilon B$ is purely log terminal for any $\epsilon > 0$ sufficiently small, replacing A by ϵA and B by $\epsilon B + (1-\epsilon)(\Gamma - T)$, we may assume that $K_Y + \Gamma = K_Y + T + A + B$.

We let primes denote restriction to T, so that, for example, $H' = H|_T$. Fix a non-zero section

$$\sigma \in H^0(T, L').$$

Let S be the zero locus of σ. By assumption, we may find a divisor $G_s \sim sc_1(L) + H$, such that

$$G'_s = sS + H'.$$

If we set

$$N = c_1(L) - K_Y - T \qquad \text{and} \qquad \Theta = \frac{m-1}{ms} G_s + B,$$

then

$$N \sim_{\mathbb{Q}} (m-1)(K_Y + \Gamma) + A + B.$$

Since

$$N - \Theta \sim_{\mathbb{Q}} A + B - \frac{m-1}{ms} H - B = A - \frac{m-1}{ms} H,$$

is ample for s sufficiently large, it follows that,

$$H^1(Y, L(-T) \otimes \mathcal{J}(\Theta)) = H^1(Y, \mathcal{O}_Y(K_Y + N) \otimes \mathcal{J}(\Theta)) = 0,$$

by Nadel vanishing (5.3). By (9.5.1) of [11] or (2.4.2) of [18], there is a short exact sequence

$$0 \longrightarrow \mathcal{J}(Y, \Theta)(-T) \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}(T, \Theta') \longrightarrow 0,$$

where $\mathcal{J}$ is an ideal sheaf. Then

(5.4.1) $$H^0(Y, L \otimes \mathcal{J}) \longrightarrow H^0(T, L' \otimes \mathcal{J}(T, \Theta')),$$

is surjective. Now

$$\begin{aligned}\Theta' - S &= B' + \frac{m-1}{ms}(sS + H') - S \\ &\leq B' + \frac{m-1}{ms}H'.\end{aligned}$$

Since $(Y, T + A + B)$ is purely log terminal, (T, B') is kawamata log terminal, and so

$$(T, B' + \frac{m-1}{ms}H'),$$

is kawamata log terminal for s sufficiently large. But then

$$\sigma \in H^0(T, L' \otimes \mathcal{J}(\Theta'))$$

by (5.2). Thus we may lift σ, using (5.4.1). □

6. Limiting algebras

To state the main result of this section, we need a:

Definition 6.1. *We say that a geometric algebra $\mathfrak{R}$, given by an additive sequence $B_\bullet$, is **limiting**, if there are $\mathbb{Q}$-divisors Δ_m and a positive integer k such that*

1. $B_m = mk(K_X + \Delta_m)$,
2. *the limit Δ of the convex sequence $\Delta_\bullet$ exists, and*
3. $K_X + \Delta$ *is kawamata log terminal.*

Theorem 6.2. *Let (X, Δ) be a log pair of dimension n and let $f \colon X \longrightarrow Z$ be a morphism, where Z is affine and normal. Let k be any positive integer such that $D = k(K_X + \Delta)$ is Cartier. Assume that*

1. $K_X + \Delta$ *is purely log terminal,*
2. $S = \llcorner \Delta \lrcorner$ *is irreducible,*
3. *there is a positive integer m_0 and a divisor $G_0 \in |m_0 D|$, such that S is not contained in the support of G_0, and*
4. $\Delta - S \sim_{\mathbb{Q}} A + B$, *where A is ample and B is an effective divisor, whose support does not contain S.*

Then there exists a birational morphism $T \longrightarrow S$ with the following properties:

For each $m > 0$, there exists a resolution $g_m \colon Y_m \longrightarrow X$, such that, if we write,

$$K_{Y_m} + \Gamma'_m = g_m^*(K_X + \Delta) + E_m,$$

where Γ'_m and E_m are effective with no common components, and we let,

(a) *N_m be the mobile part of $mk(K_{Y_m} + \Gamma'_m)$,*
(b) *$mk(K_{Y_m} + \Gamma_m)$ be the log mobile part, see (6.4), of $mk(K_{Y_m} + \Gamma'_m)$.*
(c) *$Z_m = E_m - \Gamma'_m + T$,*
(d) *$\Theta_m = (\Gamma_m - T)|_T$, and*
(e) *M_m be the mobile part of $mk(K_T + \Theta_m)$,*

then:

(i) *the strict transform of S is isomorphic, over S, to T, and $Z_m|_T$ is independent of m. Further for every m and n, there are birational morphisms $Y_{m,n} \longrightarrow Y_m$ and $Y_{m,n} \longrightarrow Y_n$, where $Y_{m,n}$ is smooth and the strict transform of T is isomorphic to T.*
(ii) *$N_m|_T = M_m$ and $|N_m|_T = |M_m|$.*
(iii) *$\Theta_\bullet$ is convex, up to truncation.*
(iv) *Let $\mathfrak{R}$ be the limiting algebra associated to $\Theta_\bullet$. Then the restricted algebra $\mathfrak{R}_S$ is finitely generated iff $\mathfrak{R}$ is finitely generated.*
(v) *If there is a positive integer s, such that the mobile part P_m of $ml(K_T + \Theta_m)$ is free, where $l = ks$, then the mobile part Q_m of $ml(K_{Y_m} + \Gamma_m)$ is free.*

Note that the hypotheses of (6.2) are simply those of (4.3), excluding (5) of (4.3), and the hypothesis that the MMP holds. To prove (6.2), we are going to apply (5.4). The idea will be to start with the main result (3.17) of [4], which we state in a convenient form:

Theorem 6.3. *Let (Y, Γ) be a smooth log pair, and let $Y \longrightarrow Z$ be a projective morphism, where Z is normal and affine. Let $H = lA$, where l is a sufficiently large and divisible positive integer and A is very ample. Let k be any positive integer such that $k(K_Y + \Gamma)$ is Cartier, and let m be any positive integer. Let L be the line bundle $\mathcal{O}_Y(mk(K_Y + \Gamma))$. Assume that*

(1) *Γ contains T with coefficient one,*
(2) *(Y, Γ) is log canonical, and*
(3) *there is a positive integer m_0 and a divisor $G_0 \in |m_0k(K_Y + \Gamma)|$ which does not contain any log canonical centre of $K_Y + \ulcorner \Gamma \urcorner$.*

Let $\Theta = (\Gamma - T)|_T$, so that

$$(K_Y + \Gamma)|_T = K_T + \Theta.$$

Suppose that H does not contain T. Then the image of the natural homomorphism

$$H^0(Y, L(H)) \longrightarrow H^0(T, L(H)|_T),$$

contains the image of $H^0(T, L|_T)$, where $H^0(T, L|_T)$ is considered as a subspace of $H^0(T, L(H)|_T)$ by the inclusion induced by $H|_T$.

Now to apply (6.3), the main point will be to change models and alter Γ, so that property (3) holds. To this end, we will need some results concerning manipulation of log pairs. Given a divisor $\Delta = \sum_i a_i \Delta_i$, we set

$$\langle \Delta \rangle = \sum_i b_i \Delta_i \qquad \text{where} \qquad b_i = \begin{cases} a_i & \text{if } 0 < a_i < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6.4. *Let (X, Δ) be a log pair and let k be a positive integer such that $k(K_X + \Delta)$ is an integral divisor. Let m be a positive integer and let $mk(K_X + \Delta) = M_m + F_m$ be the decomposition of $mk(K_X + \Delta)$ into its mobile M_m and fixed F_m part.*

*The **log mobile part** $mk(K_X + \Delta_m)$ of $mk(K_X + \Delta) = M_m + F_m$ is defined by*

$$\Delta_m = \max(\Delta - \frac{F_m}{mk}, 0),$$

*(recall that $\max(A, B)$ is defined coefficient by coefficient). We say that $mk(K_X + \Delta)$ is **log mobile**, if $\Delta = \Delta_m$*

*The sequence $\Delta_\bullet$ is called the **log mobile sequence**.*

Note that a log mobile divisor is not necessarily mobile. However, we do have:

Lemma 6.5. *Let (X, Δ) be a log pair, and let k be a positive integer such that $k(K_X + \Delta)$ is integral. Let m be a positive integer and let $mk(K_X + \Delta) = M_m + F_m$ be the decomposition of $mk(K_X + \Delta)$ into its mobile M_m and fixed F_m part.*

Then

1. *$0 \leq \Delta_m \leq \Delta$.*
2. *M_m is the mobile part of $mk(K_X + \Delta_m)$.*
3. *No component of Δ_m is a fixed component.*
4. *Possibly replacing k by a multiple, the sequence $\Delta_\bullet$ is convex.*

Proof. (1-3) are easy to check.

If M_m forms an additive sequence, then $B_m = \Delta - F_m/mk$ is a convex sequence. In this case, the only way that the sequence $\max(B_\bullet, 0)$ is not convex, is when the coefficient of a component of B_m transitions from being negative to positive. But since there are only finitely many components of Δ, this can only happen finitely many times. □

Lemma 6.6. *Let (Y, Γ) be a smooth log pair, and let $\pi\colon Y \longrightarrow Z$ be a projective morphism, where Z is normal and affine. Let k be any* *positive integer such that $k(K_Y + \Gamma)$ is integral, and let m be a positive integer. Assume that*

(1) *(Y, Γ) is purely log terminal,*
(2) *$T = \llcorner \Gamma \lrcorner$ is irreducible,*
(3) *$\Gamma - T \sim_{\mathbb{Q}} A + B$, where A is ample and B is effective, not containing T,*
(4) *no two components of $\langle \Gamma \rangle$ intersect, and*
(5) *the fixed part of $mk(K_Y + \Gamma)$ and the support of Γ has global normal crossings.*

Then there is a sequence of blow ups $h_m \colon Y_m \longrightarrow Y$ along smooth centres, which are strata of the support of the fixed locus of $mk(K_Y + \Gamma)$ and the support of Γ and which in a neighbourhood of T are smooth divisors in T, with the following property:

Suppose that we write

$$K_{Y_m} + \Gamma'_m = h_m^*(K_Y + \Gamma) + F_m.$$

Set $mk(K_{Y_m} + \Gamma_m)$ to be the log mobile part of $mk(K_{Y_m} + \Gamma'_m)$ and $\Theta_m = (\Gamma_m - T)|_T$. Let N_m be the mobile part of $mk(K_Y + \Gamma_m)$ and let M_m be the mobile part of $mk(K_T + \Theta_m)$. Then

$$N_m|_T = M_m \qquad \text{and} \qquad |N_m|_T = |M_m|.$$

Proof. Since no two components of $\langle \Gamma \rangle$ intersect, and T is the only component of coefficient one, the only possible log canonical centres of $K_Y + \ulcorner \Gamma \urcorner$ contained in the base locus of $|mk(K_Y + \Gamma)|$, are the components of $T \cap \langle \Gamma \rangle$. It follows that we may choose a sequence of blow ups $h_m \colon Y_m \longrightarrow Y$, as above, so that the base locus of N_m contains no log canonical centre of $K_{Y_m} + \ulcorner \Gamma'_m \urcorner$. But then the base locus of $mk(K_{Y_m} + \Gamma_m)$ does not contain any log canonical centre of $K_{Y_m} + \ulcorner \Gamma_m \urcorner$.

Possibly replacing kA by a linearly equivalent divisor, we may assume that $h_m^* A$ and the strict transform of A are equal. Since A is ample, there is an effective and exceptional divisor F such that $A_m = h_m^* A - F$ is ample. In this case

$$\Gamma_m - T \sim_{\mathbb{Q}} (h_m^* A - F) + (\Gamma_m - T - h_m^* A + F) \sim_{\mathbb{Q}} A_m + B_m.$$

It follows by (6.3) and (5.4), that

$$H^0(Y_m, \mathcal{O}_{Y_m}(mk(K_Y + \Gamma_m))) \longrightarrow H^0(T, \mathcal{O}_T(mk(K_T + \Theta_m))),$$

is surjective. It follows that $N_m|_T = M_m + L_m$, where L_m is contained in some fixed normal crossings divisor. Possibly blowing up more along the components of L_m, we may assume that L_m is empty. $\square$

Lemma 6.7. *Let (X,Δ) be a log pair. We may find a birational projective morphism*

$$g\colon Y \longrightarrow X,$$

with the following properties. Suppose that we write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components, and E is exceptional.

Then no two components of $\langle \Gamma \rangle$ intersect.

Proof. Passing to a log resolution, we may assume that the pair (X,Δ) has global normal crossings. We will construct g as a sequence of blow ups of irreducible components of the intersection of a collection of components of $\langle \Delta \rangle$. Now if a collection of components of $\langle \Delta \rangle$ intersect, then certainly no irreducible component of their intersection is contained in $\llcorner \Delta \lrcorner$. Since (X,Δ) has global normal crossings, it follows that we may as well replace Δ by $\langle \Delta \rangle$. Thus we may assume that the coefficients of the components of Δ are all less than one, so that the pair (X,Δ) is kawamata log terminal, and our aim is to find g, so that no two components of Γ intersect.

We proceed by induction on the maximum number k of components of Δ which intersect. Since (X,Δ) has normal crossings, $k \leq n = \dim X$, and it suffices to decrease k. We now proceed by induction on the maximum sum s of the coefficients of k components which intersect. If we pick r such that $r\Delta$ is integral, then s is at least k/r and rs is an integer, so it suffices to decrease s. We further proceed by induction on the number l of subvarieties V which are the components of the intersection of k components of Δ whose coefficients sum to s. We aim to decrease l by blowing up.

Suppose that we blow up $g\colon Y \longrightarrow X$ along the intersection V of k components $\Delta_1, \Delta_2, \dots, \Delta_k$ of Δ, with coefficients $a_1, a_2, \dots, a_k$. A simple calculation, see for example (2.29) of [10], gives that the discrepancy of the exceptional divisor E is $(k-1) - s$, so that

$$K_Y + \Gamma = g^*(K_X + \Delta) + (k - 1 - s)E.$$

If $k - 1 - s \geq 0$, then E is not a component of Γ. Otherwise E is a component of Γ, with coefficient $s + 1 - k$. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ be the components of Γ' which are the strict transforms of $\Delta_1, \Delta_2, \dots, \Delta_k$. Then there are k subvarieties of Y which dominate V, which are the intersection of k components of Γ, namely the intersection with E of all but one of $\Gamma_1, \Gamma_2, \dots, \Gamma_k$. If a_k is the smallest coefficient, then the maximum sum of the coefficients of these intersections is

$$(s - a_k) + (s + 1 - k) = s + [(s - a_k) - (k - 1)] < s,$$

and so we have decreased l by one. $\square$

Proof of (6.2). Let $g\colon Y \longrightarrow X$ be any morphism, whose existence is guaranteed by (6.7). We may write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components and E is exceptional. Since $k(K_X+\Delta)$ is Cartier, $k(K_Y+\Gamma)$ and kE are integral. Let T be the strict transform of S. Possibly blowing up more, and replacing k by a multiple, we may assume that the fixed part of $|k(K_Y+\Gamma)|$ and the support of Γ has normal crossings.

Possibly replacing kA by a linearly equivalent divisor, we may assume that g^*A and the strict transform of A are equal. Since A is ample, there is an effective and exceptional divisor F such that $A' = g^*A - F$ is ample. In this case

$$\Gamma - T \sim_{\mathbb{Q}} (g^*A - F) + (\Gamma - T - g^*A + F) \sim_{\mathbb{Q}} A' + B'.$$

Let $h_m\colon Y_m \longrightarrow Y$ be the birational morphism, whose existence is guaranteed by (6.6). Let $g_m = h_m \circ g\colon Y_m \longrightarrow X$. Then the restriction of h_m to the strict transform of T is an isomorphism, and an easy log discrepancy calculation shows that $Z_m|_T$ is independent of m. Thus (i) holds. (ii) holds by (6.6). Note that the restriction of $mk(K_Y + \Gamma_m)$ to T is equal to the restriction of the log mobile part of $mk(K_Y + \Gamma)$ to T. (iii) holds by (4) of (6.5). (iv) holds by (3.10).

Suppose that P_m is free. As $mk(K_{Y_m} + \Gamma_m)$ is log mobile, it is clear that $ml(K_{Y_m} + \Gamma_m)$ is log mobile as well. Arguing as above, and possibly blowing up more, we may assume that $Q_m|_T = P_m$ and that $|Q_m|_T = |P_m|$. In this case, Q_m is free in a neighbourhood of T, and so possibly blowing up further, we may assume that (v) holds. $\square$

7. Real versus rational

Most of the ideas and a significant part of the proofs of the results in this section are contained in [14]. We have only restated these results at the level of generality we need to prove (1.1).

We will need a generalisation of the base point free theorem to the case of real divisors:

Theorem 7.1 (Base Point Free Theorem)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial kawamata log terminal pair, where Δ is a $\mathbb{R}$-divisor. Let $f\colon X \longrightarrow Z$ be a projective morphism, where Z is affine and normal, and let D be* *a nef $\mathbb{R}$-divisor, such that $aD - (K_X + \Delta)$ is nef and big, for some positive real number a.*

Then D is semiample.

Proof. Replacing D by aD we may assume that $a = 1$. By assumption we may write

$$D - (K_X + \Delta) = A + E,$$

where A is ample and E is effective. Thus

$$D - (K_X + \Delta + \epsilon E),$$

is ample for all $\epsilon > 0$. Since the pair $(X, \Delta + \epsilon E)$ is kawamata log terminal for ϵ small enough, replacing Δ by $\Delta + \epsilon E$, we may assume that

$$D - (K_X + \Delta),$$

is ample. Perturbing Δ, we may therefore assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let F be the set of all elements α of the closed cone of curves on which D is zero. Then $K_X + \Delta$ is negative on F. Let H be any ample divisor. For every ray $R = \mathbb{R}^+\alpha$ contained in F, there is an $\epsilon > 0$ such that $K_X + \Delta + \epsilon H$ is negative on R. By compactness of a slice, it follows that there is an $\epsilon > 0$, such that $K_X + \Delta + \epsilon H$ is negative on the whole of F. It follows by the cone theorem that F is the span of finitely many extremal rays $R_1, R_2, \dots, R_k$, where each extremal ray R_i is spanned by an integral curve C_i. Let $D_1, D_2, \dots, D_l$ be the prime components of D. Consider the convex subset $\mathcal{P}$ of

$$\{\, B = \sum_i d_i D_i \,|\, d_i \in \mathbb{R} \,\},$$

consisting of all divisors B such that B is zero on F. Then $\mathcal{P}$ is a closed rational polyhedral cone.

In particular $D \in \mathcal{P}$ is a convex linear combination of divisors $B_i \in \mathcal{P} \cap N_{\mathbb{Q}} \cap U$, where U is any neighbourhood of D. But if U is sufficiently small, then $B_i - (K_X + \Delta)$ is also ample. Now if $B_i = (B_i - (K_X + \Delta)) + (K_X + \Delta)$ is not nef, then it must be negative on a $(K_X + \Delta)$-extremal ray. As the extremal rays of $K_X + \Delta$ are discrete in a neighbourhood of F, it follows that B_i is also nef if U is sufficiently small. By the base point free theorem, it follows that each B_i is semiample, so that D is semiample. □

Theorem 7.2. *Assume the real MMP in dimension n. Let (X, Δ) be a kawamata log terminal pair of dimension n, such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and big. Assume that there is a divisor Ψ such that $K_X + \Psi$ is*

$\mathbb{Q}$-Cartier and kawamata log terminal. Let $f\colon X \longrightarrow Z$ be any projective morphism, where Z is normal and affine. Fix a finite dimensional vector subspace V of the space of $\mathbb{R}$-divisors containing Δ.

Then there are finitely many birational maps $\psi_i\colon X \dashrightarrow W_i$, $1 \leq i \leq l$ over Z, such that for every divisor $\Theta \in V$ sufficiently close to Δ, there is an $1 \leq i \leq l$ with the following properties:

(1) *ψ_i is the composition of a sequence of $(K_X+\Theta)$-negative divisorial contractions and birational maps, which are isomorphisms in codimension one,*
(2) *W_i is $\mathbb{Q}$-factorial, and*
(3) *$K_{W_i} + \psi_{i*}\Theta$ is semiample.*

Further there is a positive integer s such that

(4) *if $k(K_X+\Theta)$ is integral then $sk(K_{W_i}+\psi_{i*}\Theta)$ is base point free.*

Proof. As the property of being big is an open condition, we may assume that for any $\Theta \in V$ sufficiently close to Δ, $K_X + \Theta$ is big.

Suppose that we have established (1-3). As W_i is $\mathbb{Q}$-factorial and W_i has rational singularities, it follows that the group of Weil divisors modulo Cartier divisors is a finite group. Thus there is a fixed positive integer s_i such that if $k(K_X + \Theta)$ is integral, then $s_ik(K_{W_i} + \psi_{i*}\Theta)$ is Cartier. By Kollár's effective base point free theorem, [8], there is then a positive integer M such that $Ms_ik(K_{W_i} + \psi_{i*}\Theta)$ is base point free. If we set s to be the least common multiple of the Ms_i, then this is (4). Thus it suffices to prove (1-3).

As $K_X + \Theta$ is big, if Θ is sufficiently close to Δ, by (7.1), we may replace (3) by the weaker condition,

($3'$) $K_{W_i} + \psi_{i*}\Theta$ is nef.

Thus it suffices to establish (1), (2) and ($3'$).

Since we are assuming existence and termination of flips for $\mathbb{Q}$-divisors, we may construct a log terminal model of (X, Ψ). As (X, Ψ) is kawamata log terminal, the log terminal model is small over X. Thus passing to a log terminal model of (X, Ψ), we may assume that X is $\mathbb{Q}$-factorial and that f is projective.

Suppose that $K_X + \Delta$ is not nef. Let R be an extremal ray for $K_X+\Delta$. R is necessarily $(K_X+\Theta)$-negative, for any $\mathbb{Q}$-divisor Θ close enough to Δ. By the cone and contraction theorems applied to $K_X+\Theta$, we can contract R, $\psi\colon X \longrightarrow X'$. ψ must be birational, as $K_X + \Delta$ is big. If ψ is divisorial (that is the exceptional locus is a divisor) then we replace the pair (X, Δ) by the pair $(X', \psi_*\Delta)$. If ψ is small, then using $(1.4)_{\mathbb{Q},n}$, we know the flip of $K_X+\Theta$ exists. But then this is also the flip of $K_X+\Delta$, and so we can replace the pair (X, Δ) by the flip. Since we are assuming $(1.5)_{\mathbb{R},n}$, and we can only make finitely many divisorial contractions, we must eventually arrive at the case when $K_X + \Delta$ is nef.

By (7.1) it follows that $K_X+\Delta$ is relatively semiample. Let $\psi\colon X \longrightarrow W$ be the corresponding contraction over Z. Then there is an ample $\mathbb{R}$-divisor H on W such that $K_X + \Delta = \psi^* H$. Thus if Θ is sufficiently close to Δ and $K_X + \Theta$ is relatively nef over W, then

$$K_X + \Theta = K_X + \Delta + (\Theta - \Delta) = \psi^* H + (\Theta - \Delta),$$

is nef.

Note that we may replace Z by W, and use the fact that a Cartier divisor is relatively generated iff it is locally base point free. Thus replacing Z by an open affine subset of W, we may assume that f is birational and $K_X + \Delta$ is $\mathbb{R}$-linearly equivalent to zero. Let B be the closure in V of a ball with radius δ centred at Δ. If δ is sufficiently small, then for every $\Theta \in B$, $K_X + \Theta$ is kawamata log terminal. Pick Θ a point of the boundary of B. Since $K_X + \Delta$ is $\mathbb{R}$-linearly equivalent to zero, note that for every curve C,

$$(K_X + \Theta) \cdot C < 0 \qquad \text{iff} \qquad (K_X + \Theta') \cdot C < 0, \forall \Theta' \in (\Delta, \Theta].$$

In particular every step of the $(K_X+\Theta)$-MMP is a step of $(K_X+\Theta')$-MMP, for every $\Theta' \in (\Delta, \Theta]$. Since we are assuming existence and termination of flips, we have a birational map $\psi\colon X \dashrightarrow W$ over Z, such that $K_W + \psi_*\Theta$ is nef, and it is clear that $K_W + \psi_*\Theta'$ is nef, for every $\Theta' \in (\Delta, \Theta]$.

At this point we want to proceed by induction on the dimension of B. To this end, note that as B is compact and Δ is arbitrary, our result is equivalent to proving that $(3')$ holds in B. By what we just said, this is equivalent to proving that $(3')$ holds on the boundary of B, which is a compact polyhedral cone (since we are working in the sup norm) and we are done by induction on the dimension of B. ☐

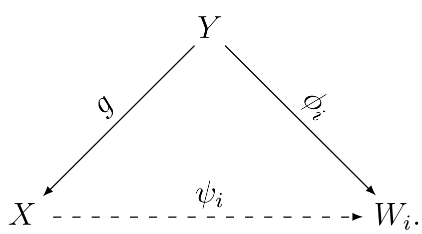
Here is the key consequence of (7.2):

Corollary 7.3. *Assume the real MMP in dimension n. Let (X, Δ) be a kawamata log terminal $\mathbb{Q}$-factorial pair of dimension n, where $K_X + \Delta$ is an $\mathbb{R}$-divisor. Let $f\colon X \longrightarrow Z$ be a contraction morphism. Let k be a positive integer. Fix a finite dimensional vector subspace V of the space of $\mathbb{R}$-divisors containing Δ.*

If $K_X + \Delta$ is relatively big, then there is a birational model $g\colon Y \longrightarrow X$ and a positive integer s, such that if $\pi\colon Y \longrightarrow Z$ is the composition of f and g, then for every divisor $\Theta \in V$ sufficiently close to Δ,

(1) *if* $k(K_X+\Theta)$ *is integral, then the mobile part of* $g^*(ks(K_X+\Theta))$ *is base point free, where* $l=ks$.
(2) *If* $\Theta_\bullet$ *is a convex sequence of divisors with limit* Θ, *such that* $mk(K_X+\Theta_m)$ *is integral, then the limit* D *of the characteristic sequence* $D_\bullet$ *associated to* $B_m = g^*(ml(K_X+\Theta_m))$ *is semiample.*

Proof. Let $\psi_i\colon X \dashrightarrow W_i$ be the models, whose existence is guaranteed by (7.2), and let $g\colon Y \longrightarrow X$ be any birational morphism which resolves the indeterminancy of ψ_i, $1 \leq i \leq l$. Let $\phi_i\colon Y \longrightarrow W_i$ be the induced birational morphisms, so that we have commutative diagrams



Let Θ be sufficiently close to Δ. Then for some i, $K_{W_i}+\psi_{i*}\Theta$ is semiample. Suppressing the index i, we may write

$$g^*(K_X+\Theta) = \phi^*(K_W+\psi_*\Theta)+E+F,$$

where for E we sum over the common exceptional divisors of g and ϕ, and for F we sum over the exceptional divisors of ϕ which are not exceptional for g (by assumption there are no exceptional divisors of g which are not also exceptional for ϕ). (1) of (7.2) implies that F is effective. But then by negativity of contraction, see (2.19) of [9], E is also effective.

Suppose that $k(K_X+\Theta)$ is integral. By (7.2) there is a fixed s such that $l(K_W+\psi_*\Theta)$ is free. In this case the mobile part of $g^*(l(K_X+\Theta))$ is equal to the mobile part of $\phi^*(l(K_W+\psi_*\Theta))$, which is free. This is (1).

Now suppose that $\Theta_\bullet$ is a convex sequence with limit Θ, such that $mk(K_X+\Theta_m)$ is integral. Let M_m be the mobile part of $g^*(ml(K_X+\Theta_m))$. Then, by what we have already said, M_m is also the mobile part of $\phi_i^*(ml(K_W+\psi_{i*}\Theta_m))$. Possibly passing to a subsequence, we may assume that i is constant, and in this case we suppress it. It is then clear that the limit D of

$$D_m = \frac{M_m}{m},$$

is

$$\phi^*(l(K_W+\psi_*\Theta)),$$

so that D is nef. It follows that D is semiample by (7.2) (or indeed (7.1)). □

8. Diophantine Approximation

All of the results in this section are implicit in the work of Shokurov [15], and we claim no originality. In fact we have only taken Corti's excellent introduction to Shokurov's work on the existence of flips and restated those results without the use of b-divisors.

Lemma 8.1 (Diophantine Approximation)**.** *Let Y be a smooth variety and let $\pi\colon Y \longrightarrow Z$ be a projective morphism, where Z is affine and normal. Let D be a semiample divisor on Y. Let $\epsilon > 0$ be a positive rational number.*

Then there is an integral divisor M and a positive integer m such that

1. *M is base point free,*
2. *$\|mD - M\| < \epsilon$, and*
3. *If $mD \geq M$ then $mD = M$.*

Proof. If D is rational, then pick m such that mD is integral and set $M = mD$. Thus we may suppose that D is not rational.

Let $N_{\mathbb{Z}}$ be the lattice spanned by the components G_j of D, and let $N_{\mathbb{Q}}$ and $N_{\mathbb{R}}$ be the corresponding vector spaces. Since D is semiample and π is projective, we may pick a basis $\{P_k\}$ of $N_{\mathbb{Q}}$, where each P_k is base point free, and D belongs to the cone

$$\mathcal{P} = \sum \mathbb{R}_+[P_k] \subset \mathbb{R}_+[G_k] = \mathcal{G}.$$

Let $v \in N_{\mathbb{R}}$ be the vector corresponding to D. Let A be the cyclic subgroup of the torus

$$\frac{N_{\mathbb{R}}}{N_{\mathbb{Z}}},$$

generated by the image of v. Let $\bar{A}$ be the closure of A and let A_0 be the connected component of the identity of $\bar{A}$. Let $V \subset N_{\mathbb{R}}$ be the inverse image of A_0. Then A_0 is a Lie group and so V is a linear subspace. As we are assuming that D is not rational, A is infinite and so A_0 and V are both positive dimensional. In particular V is not contained in $\mathcal{G}$. But then for every $\epsilon > 0$, we can find a positive multiple mv of v, and a vector $w \in N_{\mathbb{Z}}$, which is an integral linear combination of the divisors P_k, such that

- $\|mv - w\| < \epsilon$, whilst
- $mv - w \notin \mathcal{G}$.

Note that if $\epsilon > 0$ is sufficiently small then $w \in \mathcal{P}$ since it is integral and close to $mv \in \mathcal{P}$. Thus if M is the divisor corresponding to w, then M is base point free, and the rest is clear. □

Definition 8.2. *Let $\pi\colon Y \longrightarrow Z$ be a projective morphism of normal varieties, where Z is affine. Let $\mathfrak{R}$ be the geometric algebra associated to the additive sequence $M_\bullet$ of mobile divisors, with characteristic sequence $D_\bullet$.*

*We say that $\mathfrak{R}$ is **saturated** if there is a $\mathbb{Q}$-divisor F such that*

1. *$\ulcorner F \urcorner \geq 0$, and*
2. *for every pair of integers i and j,*

$$\operatorname{Mob}(\ulcorner jD_i + F \urcorner) \leq M_j.$$

Theorem 8.3. *Let Y be a smooth variety and $\pi\colon Y \longrightarrow Z$ a projective morphism, where Z is affine and normal. Let $\mathfrak{R}$ be a saturated semiample geometric algebra on Y.*

Then $\mathfrak{R}$ is finitely generated.

Proof. Let $D_\bullet$ be the characteristic sequence, with limit D. Let G be the support of D, and pick $\epsilon > 0$ such that $\ulcorner F - \epsilon G \urcorner \geq 0$. By diophantine approximation, we know that there is a positive integer m and an integral divisor M such that

1. M is mobile,
2. $\|mD - M\| < \epsilon$, and
3. If $mD \geq M$ then $mD = M$.

But then

$$\begin{aligned} mD + F &= M + (mD - M) + F \\ &\geq M + F - \epsilon G, \end{aligned}$$

so that

$$\operatorname{Mob}(\ulcorner mD + F \urcorner) \geq M.$$

On the other hand, by definition of saturation we have

$$\operatorname{Mob}(\ulcorner mD_i + F \urcorner) \leq M_m = mD_m.$$

Letting i go to infinity we have

$$M \leq \operatorname{Mob}(\ulcorner mD + F \urcorner) \leq mD_m \leq mD.$$

By (3) above, it follows that the sequence of inequalities must in fact be equalities, so that we have

$$D = D_m,$$

for some m and we may apply (3.11). □

9. Saturation on T

We fix some notation for this section. Let (X, Δ) be a purely log terminal pair and let $f\colon X \longrightarrow Z$ be a projective morphism of normal varieties, where Z is affine. We assume that $S = \llcorner \Delta \lrcorner$ is irreducible. Let $g_m\colon Y_m \longrightarrow X$ be the birational morphisms whose existence is guaranteed by (6.2). Then we may write

$$K_{Y_m} + \Gamma'_m = g_m^*(K_X + \Delta) + E_m,$$

where Γ'_m and E_m are effective, with no common components and E_m is g_m-exceptional. Let k be a positive integer such that $k(K_X + \Delta)$ is Cartier, and assume that the fixed part of $k(K_X+\Delta)$ does not contain S. Then both $k(K_{Y_m} + \Gamma'_m)$ and kE_m are integral. Let $mk(K_{Y_m} + \Gamma_m)$ be the log mobile part of $mk(K_{Y_m} + \Gamma'_m)$. Set $T = \llcorner \Gamma_m \lrcorner$ the strict transform of S, $Z_m = E_m - \Gamma'_m + T$ and $Z = Z_m|_T$ (note that T and Z do not depend on m, by (6.2)). As $K_X + \Delta$ is purely log terminal, $\ulcorner Z_m \urcorner$ is effective and exceptional. Let $\Theta_m = (\Gamma_m - T)|_T$ and let M_m be the mobile part of $mk(K_T + \Theta_m)$. Our aim is to prove that the algebra $\mathcal{R}$ associated to $mk(K_T + \Theta_m)$ is finitely generated.

By (7.3), we may assume that there is a birational model $h\colon T' \longrightarrow T$ of T and a positive integer s, such that the mobile part of $h^*(ml(K_T + \Theta_m))$ is free. Possibly blowing up more, we may therefore assume that the mobile part P_m of $ml(K_T + \Theta_m)$ is free, where $l = ks$. Let Q_m be the mobile part of $ml(K_Y + \Gamma_m)$. By (7.3) we may assume that Q_m is free. Note that

$$sM_m \leq P_m \leq M_{ms}.$$

Let $D_\bullet$ be the characteristic sequence of $M_\bullet$, let $J_\bullet$ be the characteristic sequence of $Q_\bullet$, and let $I_\bullet$ be the characteristic sequence of $P_\bullet$.

Lemma 9.1. *For every pair of positive integers i and j,*

$$\operatorname{Mov}(\ulcorner jJ_i + Z_i \urcorner)|_T \leq M_{js}.$$

Proof. We work on the common resolution $Y_{i,js}$ of Y_i and Y_{js}, whose existence is guaranteed by (1) of (6.2). We have

$$\begin{aligned}\mathrm{Mov}(\ulcorner jJ_i + Z_i \urcorner) &\leq \mathrm{Mov}(\ulcorner jl(K_{Y_i} + \Gamma_i) + Z_i \urcorner)\\ &\leq \mathrm{Mov}(\ulcorner jl(K_{Y_i} + \Gamma'_i) + Z_i \urcorner)\\ &= \mathrm{Mov}(jl(K_{Y_i} + \Gamma'_i) + \ulcorner Z_i \urcorner)\\ &= \mathrm{Mov}(jlg_i^*(K_X + \Delta) + jlE_i + \ulcorner Z_i \urcorner)\\ &= \mathrm{Mov}(jlg_i^*(K_X + \Delta))\\ &\leq \mathrm{Mov}(jlg_{js}^*(K_X + \Delta) + jlE_{js})\\ &= \mathrm{Mov}((js)k(K_{Y_{js}} + \Gamma'_{js})) = N_{js},\end{aligned}$$

where we used the fact that $jlE_i + \ulcorner Z_i \urcorner$ is effective and g_i-exceptional. Restricting to T we get

$$\mathrm{Mov}(\ulcorner jJ_i + Z_i \urcorner)|_T \leq N_{js}|_T = M_{js}.$$

□

Lemma 9.2. *The natural restriction map,*

$$H^0(Y_i, \mathcal{O}_{Y_i}(\ulcorner jJ_i + Z_i \urcorner)) \longrightarrow H^0(T, \mathcal{O}_T(\ulcorner jI_i + Z \urcorner)),$$

is surjective, for any positive integers i and j.

Proof. Considering the restriction exact sequence,

$$0 \longrightarrow \mathcal{O}_{Y_i}(\ulcorner jJ_i + Z_i \urcorner - T) \longrightarrow \mathcal{O}_{Y_i}(\ulcorner jJ_i + Z_i \urcorner) \longrightarrow \mathcal{O}_T(\ulcorner jI_i + Z \urcorner) \longrightarrow 0,$$

it follows that the obstruction to surjectivity of the restriction map above is given by,

$$\begin{aligned}&H^1(Y_i, \mathcal{O}_{Y_i}(\ulcorner jJ_i + (Z_i - T) \urcorner))\\ &\qquad = H^1(Y_i, \mathcal{O}_{Y_i}(K_{Y_i} + \ulcorner g_i^*(-(K_X + \Delta) + jJ_i) \urcorner)),\end{aligned}$$

which vanishes by Kawamata-Viehweg vanishing, as

$$g_i^*(-(K_X + \Delta)),$$

is big and nef and jJ_i is nef. □

Lemma 9.3. *Suppose that $-(K_X + \Delta)$ is nef and big. Then the truncation of algebra $\mathfrak{R}$, given by s, is saturated with respect to Z.*

Proof. By assumption

$$\ulcorner Z \urcorner \geq 0.$$

On the other hand, by (9.2),

$$\mathrm{Mov}(\ulcorner jI_i + Z \urcorner) \leq \mathrm{Mov}(\ulcorner jJ_i + Z_i \urcorner)|_T.$$

Thus

$$\begin{aligned}\operatorname{Mov}(\ulcorner (js)D_i + Z\urcorner) &\leq \operatorname{Mov}(\ulcorner jI_i + Z\urcorner)\\ &\leq M_{js} = jsD_{js},\end{aligned}$$

where we used (9.1), and the fact that $sM_i \leq P_i$. If we replace i by is and then pass from the sequence D_i to the sequence D_{is}, this has the effect of replacing js by j, and the result follows. □

Proof of (4.3). By (iv) of (6.2), it suffices to prove that $\mathfrak{R}$ is finitely generated. We want to apply (8.3). By (9.3), we may assume that $\mathfrak{R}$ is saturated. By (7.3), the limit I of the characteristic sequence $I_{\bullet}$ is semiample. On the other hand, as

$$sD_i \leq I_i \leq sD_{si},$$

$I = sD$, so that D is also semiample, and we may apply (8.3). □

Proof of (1.1). Immediate from (4.3) and (4.4). □

Proof of (1.2). Clear. □

Proof of (1.3). Follows from (5.1.3) of [14]. □

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